K-GROUPS FOR RINGS OF FINITE COHEN–MACAULAY TYPE

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Abstract. For a local Cohen–Macaulay ring R of finite CM-type, Yoshino has applied methods of Auslander and Reiten to compute the Grothendieck group K_0 of the category $\mathsf{mod}\,R$ of finitely generated R-modules. For the same type of rings, we compute in this paper the first Quillen K-group $\mathrm{K}_1(\mathsf{mod}\,R)$. We also describe the group homomorphism $R^* \to \mathrm{K}_1(\mathsf{mod}\,R)$ induced by the inclusion functor $\mathsf{proj}\,R \to \mathsf{mod}\,R$ and illustrate our results with concrete examples.

1. Introduction

Throughout this introduction, R denotes a commutative noetherian local Cohen–Macaulay ring. The lower K-groups of R are known: $\mathrm{K}_0(R) \cong \mathbb{Z}$ and $\mathrm{K}_1(R) \cong R^*$. For $n \in \{0,1\}$ the classical K-group $\mathrm{K}_n(R)$ of the ring coincides with Quillen's K-group $\mathrm{K}_n(\mathsf{proj}\,R)$ of the exact category of finitely generated projective R-modules; and if R is regular, then Quillen's resolution theorem shows that the inclusion functor $\mathsf{proj}\,R \to \mathsf{mod}\,R$ induces an isomorphism $\mathrm{K}_n(\mathsf{proj}\,R) \cong \mathrm{K}_n(\mathsf{mod}\,R)$. If R is non-regular, then these groups are usually not isomorphic. The groups $\mathrm{K}_n(\mathsf{mod}\,R)$ are often denoted $\mathrm{G}_n(R)$ and they are classical objects of study called the G-theory of R. A celebrated result of Quillen is that G-theory is well-behaved under (Laurent) polynomial extensions: $\mathrm{G}_n(R[t]) \cong \mathrm{G}_n(R)$ and $\mathrm{G}_n(R[t,t^{-1}]) \cong \mathrm{G}_n(R) \oplus \mathrm{G}_{n-1}(R)$.

Auslander and Reiten [3] showed how to compute $\mathrm{K}_0(\mathsf{mod}\,\Lambda)$ for an Artin algebra Λ of finite representation type. Using similar techniques, Yoshino [21] computed $\mathrm{K}_0(\mathsf{mod}\,R)$ in the case where R has finite (as opposed to tame or wild) CM-type:

Theorem (Yoshino [21, thm. (13.7)]). *Assume that R is henselian and that it has a dualizing module. If R has finite CM-type, then there is a group isomorphism,*

$$\mathrm{K}_0(\mathsf{mod}\,R) \cong \operatorname{Coker}\Upsilon\ ,$$

where $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ is the Auslander–Reiten homomorphism from (2.2).

Yoshino's result is as much a contribution to algebraic K-theory as it is to the representation theory of the category $\mathsf{MCM}\,R$ of maximal Cohen–Macaulay R-modules. Indeed, for every integer $n \geqslant 0$ the inclusion functor $\mathsf{MCM}\,R \to \mathsf{mod}\,R$ induces a group isomorphism $\mathrm{K}_n(\mathsf{MCM}\,R) \cong \mathrm{K}_n(\mathsf{mod}\,R)$.

In this paper, we build upon results and techniques of Auslander and Reiten [3], Bass [6], Lam [12], Leuschke [13], Quillen [15], Vaserstein [19, 20], and Yoshino [21] to compute the group $\mathrm{K}_1(\mathsf{mod}\,R)$ when R has finite CM-type. Our main result is Theorem (2.11); it asserts that there is an isomorphism,

$$\mathrm{K}_1(\mathsf{mod}\,R) \cong \operatorname{Aut}_R(M)_{\mathrm{ab}}/\Xi\ ,$$

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where M is any representation generator of the category of maximal Cohen–Macaulay R-modules and $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ is the abelianization of its automorphism group. The subgroup Ξ is more complicated to describe; it is determined by the Auslander–Reiten sequences and defined in (2.9). We also show that if one writes $M = R \oplus M'$, then the homomorphism $R^* \cong \mathrm{K}_1(\mathsf{proj}\, R) \to \mathrm{K}_1(\mathsf{mod}\, R)$ induced by the inclusion functor $\mathsf{proj}\, R \to \mathsf{mod}\, R$ can be identified with the map

$$\lambda\colon R^* \longrightarrow \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi \qquad \text{given by} \qquad r \longmapsto \begin{pmatrix} r1_R & 0 \\ 0 & 1_{M'} \end{pmatrix}.$$

The paper is organized as follows: In Section 2 we formulate our main result, Theorem (2.11). This theorem is not proved until Section 7, and the intermediate Sections 3 (on Auslander's and Reiten's theory for coherent pairs), 4 (on Vaserstein's result for semilocal rings), 5 (on some useful equivalences of categories), and 6 (on Yoshino's results for the abelian category $\mathcal{Y}$) prepare the ground.

In Sections 8 and 9 we apply our main theorem to compute the group $\mathrm{K}_1(\mathsf{mod}\, R)$ and the homomorphism $\lambda\colon R^* \to \mathrm{K}_1(\mathsf{mod}\, R)$ in some concrete examples. E.g. for the simple curve singularity $R = k[\![T^2,T^3]\!]$ we obtain $\mathrm{K}_1(\mathsf{mod}\, R) \cong k[\![T]\!]^*$ and show that the homomorphism $\lambda\colon k[\![T^2,T^3]\!]^* \to k[\![T]\!]^*$ is the inclusion. It is well-known that if R is artinian with residue field k, then one has $\mathrm{K}_1(\mathsf{mod}\, R) \cong k^*$. We apply Theorem (2.11) to confirm this isomorphism for the ring $R = k[X]/(X^2)$ of dual numbers and to show that the homomorphism $\lambda\colon R^* \to k^*$ is given by $a+bX \mapsto a^2$; this is a special case of Proposition (9.4), which might be well-known to experts.

The paper ends with an appendix. In the proof of Theorem (2.11) we compare and identify various K-groups. Appendix A is devoted to the Gersten–Sherman transformation, which is a natural transformation $\zeta\colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1$ where $\mathrm{K}_1^{\mathrm{B}}$ is Bass' K_1-functor for skeletally small exact categories. Examples by Gersten and Murthy show that for a general exact category $\mathcal{C}$ the homomorphism $\zeta_{\mathcal{C}}\colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C})$ need not be an isomorphism; in fact, $\zeta_{\mathsf{mod}\,\mathbb{Z}C_2}$ is not surjective. On the other hand, it is crucial for our proof of Theorem (2.11) that $\zeta_{\mathsf{mod}\, R}$ is an isomorphism in the case where R has finite CM-type.

2. Formulation of the Main Theorem

Let R be a commutative noetherian local Cohen–Macaulay ring. By $\mathsf{mod}\, R$ we denote the abelian category of finitely generated R-modules. The exact categories of finitely generated projective modules and of maximal Cohen–Macaulay modules over R are written $\mathsf{proj}\, R$ and $\mathsf{MCM}\, R$, respectively. The goal of this section is to state our main Theorem (2.11); its proof is postponed to Section 7.

(2.1) **Setup.** Throughout this paper, $(R,\mathfrak{m},k)$ is a commutative noetherian local Cohen–Macaulay ring satisfying the following assumptions.

(1) R is henselian.

(2) R admits a dualizing module.

(3) R has *finite CM-type*, that is, up to isomorphism, there are only finitely many non-isomorphic indecomposable maximal Cohen–Macaulay R-modules.

Note that (1) and (2) hold if R is $\mathfrak{m}$-adically complete. Since R is henselian, the category $\mathsf{mod}\, R$ is Krull–Schmidt by [21, prop. (1.18)]; this fact will be important a number of times in this paper.

Set $M_0 = R$ and let $M_1, \ldots, M_t$ be a set of representatives for the isomorphism classes of non-free indecomposable maximal Cohen–Macaulay R-modules. Let M be any *representation generator* of $\mathsf{MCM}\, R$—that is, a finitely generated R-module such that $\mathsf{add}_R M = \mathsf{MCM}\, R$ holds—for example, M could be $M_0 \oplus M_1 \oplus \cdots \oplus M_t$. We denote by $E = \mathrm{End}_R(M)$ the endomorphism ring of M.

It follows from [21, thm. (4.22)] that R is an isolated singularity, and hence by *loc. cit.* thm. (3.2) the category $\mathsf{MCM}\, R$ admits Auslander–Reiten sequences. Let

$$0 \longrightarrow \tau(M_j) \longrightarrow X_j \longrightarrow M_j \longrightarrow 0 \qquad (1 \leqslant j \leqslant t) \tag{2.1.1}$$

be the Auslander–Reiten sequence in $\mathsf{MCM}\, R$ ending in M_j, where τ is the Auslander–Reiten translation.

(2.2) **Definition.** For each Auslander–Reiten sequence (2.1.1) we have

$$X_j \cong M_0^{n_{0j}} \oplus M_1^{n_{1j}} \oplus \cdots \oplus M_t^{n_{tj}}$$

for uniquely determined $n_{0j}, n_{1j}, \ldots, n_{tj} \geqslant 0$. Consider the element,

$$\tau(M_j) + M_j - n_{0j} M_0 - n_{1j} M_1 - \cdots - n_{tj} M_t \;,$$

in the free abelian group $\mathbb{Z}M_0 \oplus \mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t$, and write this element as,

$$y_{0j} M_0 + y_{1j} M_1 + \cdots + y_{tj} M_t \;,$$

where $y_{0j}, y_{1j}, \ldots, y_{tj} \in \mathbb{Z}$. Define the *Auslander–Reiten matrix* Υ as the $(t+1) \times t$ matrix with entries in $\mathbb{Z}$ whose j'th column is $(y_{0j}, y_{1j}, \ldots, y_{tj})$. When Υ is viewed as a homomorphism of abelian groups $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ (elements in $\mathbb{Z}^t$ and $\mathbb{Z}^{t+1}$ are viewed as column vectors), we refer to it as the *Auslander–Reiten homomorphism.*

(2.3) **Example.** Let $R = \mathbb{C}[\![X,Y,Z]\!]/(X^3 + Y^4 + Z^2)$. Besides $M_0 = R$ there are exactly $t = 6$ non-isomorphic indecomposable maximal Cohen–Macaulay modules, and the Auslander–Reiten sequences have the following form,

$$\begin{gathered}
0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow 0 \\
0 \longrightarrow M_2 \longrightarrow M_1 \oplus M_3 \longrightarrow M_2 \longrightarrow 0 \\
0 \longrightarrow M_3 \longrightarrow M_2 \oplus M_4 \oplus M_6 \longrightarrow M_3 \longrightarrow 0 \\
0 \longrightarrow M_4 \longrightarrow M_3 \oplus M_5 \longrightarrow M_4 \longrightarrow 0 \\
0 \longrightarrow M_5 \longrightarrow M_4 \longrightarrow M_5 \longrightarrow 0 \\
0 \longrightarrow M_6 \longrightarrow M_0 \oplus M_3 \longrightarrow M_6 \longrightarrow 0 \;;
\end{gathered}$$

see [21, (13.9)]. The 7×6 Auslander–Reiten matrix Υ is therefore given by

$$\Upsilon = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}.$$

In this case, the Auslander–Reiten homomorphism $\Upsilon\colon \mathbb{Z}^6 \to \mathbb{Z}^7$ is clearly injective.

One hypothesis in our main result, Theorem (2.11) below, is that the Auslander–Reiten homomorpism Υ over the ring R in question is injective. We are not aware of an example where Υ is not injective. The following lemma covers the situation of

the rational double points, that is, the fixed point rings $R = k[\![X,Y]\!]^G$, where k is an algebraically closed field of characteristic 0 and G is a non-trivial finite subgroup of $\mathrm{SL}_2(k)$; see [4].

(2.4) **Lemma.** *Assume that R is complete, integrally closed, non-regular, Gorenstein, of Krull dimension 2, and that the residue field k is algebraically closed. Then the Auslander–Reiten homomorphism Υ is injective.*

Proof. Let $1 \leqslant j \leqslant t$ be given and consider the expression

$$\tau(M_j) + M_j - n_{0j}M_0 - n_{1j}M_1 - \cdots - n_{tj}M_t \;=\; y_{0j}M_0 + y_{1j}M_1 + \cdots + y_{tj}M_t$$

in the free abelian group $\mathbb{Z}M_0 \oplus \mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t$, see Definition (2.2). Let Γ be the Auslander–Reiten quiver of $\mathsf{MCM}\,R$. We recall from [4, thm. 1] that the arrows in Γ occur in pairs $\circ \rightleftarrows \circ$, and that collapsing each pair to an undirected edge gives an extended Dynkin diagram $\tilde{\Delta}$. Moreover, removing the vertex corresponding to $M_0 = R$ and any incident edges gives a Dynkin graph Δ.

Now, X_j has a direct summand M_k if and only if there is an arrow $M_k \to M_j$ in Γ. Also, the Auslander–Reiten translation τ satisfies $\tau(M_j) = M_j$ by [4, proof of thm. 1]. Combined with the structure of the Auslander–Reiten quiver, this means that

$$y_{kj} = \begin{cases} \;2 & \text{if } k = j\ , \\ -1 & \text{if there is an edge } M_k \text{ —— } M_j \text{ in } \tilde{\Delta}\ , \\ \;0 & \text{otherwise}\ . \end{cases}$$

Hence the $t \times t$ matrix Υ_0 with $(y_{1j}, \ldots, y_{tj})$ as j'th column, where $1 \leqslant j \leqslant t$, is the Cartan matrix of the Dynkin graph Δ; cf. [7, def. 4.5.3]. This matrix is invertible by [9, exer. (21.18)]. Deleting the first row $(y_{01}, \ldots, y_{0t})$ in the Auslander–Reiten matrix Υ, we get the invertible matrix Υ_0, and consequently, $\Upsilon\colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ determines an injective homomorphism. $\square$

For a group G we denote by G_{ab} its *abelianization*, i.e. $G_{\mathrm{ab}} = G/[G,G]$, where $[G,G]$ is the commutator subgroup of G.

We refer to the following as the *tilde construction*. It associates to every automorphism $\alpha\colon X \to X$ of a maximal Cohen–Macaulay module X an automorphism $\tilde{\alpha}\colon M^q \to M^q$ of the smallest power q of the representation generator M such that X is a direct summand of M^q.

(2.5) **Construction.** The chosen representation generator M for $\mathsf{MCM}\,R$ has the form $M = M_0^{m_0} \oplus \cdots \oplus M_t^{m_t}$ for uniquely determined integers $m_0, \ldots, m_t > 0$. For any module $X = M_0^{n_0} \oplus \cdots \oplus M_t^{n_t}$ in $\mathsf{MCM}\,R$, we define natural numbers,

$$\begin{aligned} q &= q(X) = \min\{p \in \mathbb{N} \,|\, pm_j \geqslant n_j \text{ for all } 0 \leqslant j \leqslant t\}\ , \text{ and} \\ v_j &= v_j(X) = qm_j - n_j \geqslant 0\ , \end{aligned}$$

and a module $Y = M_0^{v_0} \oplus \cdots \oplus M_t^{v_t}$ in $\mathsf{MCM}\,R$. Let $\psi\colon X \oplus Y \xrightarrow{\cong} M^q$ be the R-isomorphism that maps an element

$$((\underline{x}_0, \ldots, \underline{x}_t), (\underline{y}_0, \ldots, \underline{y}_t)) \in X \oplus Y = (M_0^{n_0} \oplus \cdots \oplus M_t^{n_t}) \oplus (M_0^{v_0} \oplus \cdots \oplus M_t^{v_t})\ ,$$

where $\underline{x}_j \in M_j^{n_j}$ and $\underline{y}_j \in M_j^{v_j}$, to the element

$$((\underline{z}_{01}, \ldots, \underline{z}_{t1}), \ldots, (\underline{z}_{0q}, \ldots, \underline{z}_{tq})) \in M^q = (M_0^{m_0} \oplus \cdots \oplus M_t^{m_t})^q\ ,$$

where $\underline{z}_{j1}, \ldots, \underline{z}_{jq} \in M_j^{m_j}$ are given by $(\underline{z}_{j1}, \ldots, \underline{z}_{jq}) = (\underline{x}_j, \underline{y}_j) \in M_j^{qm_j} = M_j^{n_j + v_j}$.

Now, given α in $\mathrm{Aut}_R(X)$, we define $\tilde{\alpha}$ to be the uniquely determined element in $\mathrm{Aut}_R(M^q)$ that makes the following diagram commutative,

$$\begin{tikzcd} X \oplus Y \arrow[r, "\psi", "\cong"'] \arrow[d, "\alpha\oplus 1_Y"', "\cong"] & M^q \arrow[d, "\cong"', "\tilde{\alpha}"] \\ X \oplus Y \arrow[r, "\cong", "\psi"'] & M^q\,. \end{tikzcd}$$

The automorphism $\tilde{\alpha}$ of M^q has the form $\tilde{\alpha} = (\tilde{\alpha}_{ij})$ for uniquely determined endomorphisms $\tilde{\alpha}_{ij}$ of M, that is, $\tilde{\alpha}_{ij} \in E = \mathrm{End}_R(M)$. Hence $\tilde{\alpha} = (\tilde{\alpha}_{ij})$ can naturally be viewed as an invertible $q \times q$ matrix with entries in E.

(2.6) **Example.** Let $M = M_0 \oplus \cdots \oplus M_t$ and $X = M_j$. Then $q = 1$ and

$$Y = M_0 \oplus \cdots \oplus M_{j-1} \oplus M_{j+1} \oplus \cdots \oplus M_t\ .$$

The isomorphism $\psi\colon X \oplus Y \to M$ maps $(x_j, (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_t))$ in $X \oplus Y$ to $(x_0, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_t)$ in M. Therefore, for $\alpha \in \mathrm{Aut}_R(X) = \mathrm{Aut}_R(M_j)$, Construction (2.5) yields the following automorphism of M,

$$\tilde{\alpha} = \psi(\alpha \oplus 1_Y)\psi^{-1} = \begin{pmatrix} 1_{M_0} & & & & & & \\ & \ddots & & & & & \\ & & 1_{M_{j-1}} & & & & \\ & & & \alpha & & & \\ & & & & 1_{M_{j+1}} & & \\ & & & & & \ddots & \\ & & & & & & 1_{M_t} \end{pmatrix},$$

which is an invertible 1×1 (block) matrix with entry in $E = \mathrm{End}_R(M)$.

The following result on Auslander–Reiten sequences is quite standard. We provide a few proof details along with the appropriate references.

(2.7) **Proposition.** *Let there be given Auslander–Reiten sequences in* $\mathsf{MCM}\,R$*,*

$$0 \longrightarrow \tau(M) \longrightarrow X \longrightarrow M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \tau(M') \longrightarrow X' \longrightarrow M' \longrightarrow 0\ .$$

If $\alpha\colon M \to M'$ *is a homomorphism then there exist homomorphisms* β *and* γ *that make the following diagram commutative,*

$$\begin{tikzcd} 0 \arrow[r] & \tau(M) \arrow[r] \arrow[d, dashed, "\gamma"] & X \arrow[r] \arrow[d, dashed, "\beta"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & \tau(M') \arrow[r] & X' \arrow[r] & M' \arrow[r] & 0\,. \end{tikzcd}$$

Furthermore, if α *is an isomorphism then so are* β *and* γ*.*

Proof. Write $\rho\colon X \to M$ and $\rho'\colon X' \to M'$. It suffices to prove the existence of β such that $\rho'\beta = \alpha\rho$, because then the existence of γ follows from diagram chasing.

As $0 \to \tau(M') \to X' \to M' \to 0$ is an Auslander–Reiten sequence, it suffices by [21, lem. (2.9)] to show that $\alpha\rho\colon X \to M'$ is not a split epimorphism. Suppose that there do exist $\tau\colon M' \to X$ with $\alpha\rho\tau = 1_{M'}$. Hence α is a split epimorphism. As M is indecomposable, α must be an isomorphism. Thus $\rho\tau\alpha = \alpha^{-1}(\alpha\rho\tau)\alpha = 1_M$, which contradicts the fact that ρ is not a split epimorphism.

The fact that β and γ are isomorphisms if α is so follows from [21, lem. (2.4)]. $\square$

The choice requested in the following construction is possible by Proposition (2.7).

(2.8) **Construction.** Choose for each $1 \leqslant j \leqslant t$ and every $\alpha \in \mathrm{Aut}_R(M_j)$ elements $\beta_{j,\alpha} \in \mathrm{Aut}_R(X_j)$ and $\gamma_{j,\alpha} \in \mathrm{Aut}_R(\tau(M_j))$ that make the next diagram commute,

$$
\begin{tikzcd}
0 \arrow[r] & \tau(M_j) \arrow[r] \arrow[d, "\cong"', "\gamma_{j,\alpha}"] & X_j \arrow[r] \arrow[d, "\cong"', "\beta_{j,\alpha}"] & M_j \arrow[r] \arrow[d, "\cong"', "\alpha"] & 0 \\
0 \arrow[r] & \tau(M_j) \arrow[r] & X_j \arrow[r] & M_j \arrow[r] & 0\,;
\end{tikzcd}
\tag{2.8.1}
$$

here the row(s) is the j'th Auslander–Reiten sequence (2.1.1).

As shown in Lemma (4.1), the endomorphism ring $E = \mathrm{End}_R(M)$ of the chosen representation generator M is semilocal, that is, $E/\mathrm{J}(E)$ is semisimple. Thus, if the ground ring R, and hence also the endomorphism ring E, is an algebra over the residue field k and $\mathrm{char}(k) \neq 2$, then a result by Vaserstein [20, thm. 2] yields that the canonical homomorphism $\theta_E \colon E^*_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{C}}(E)$ is an isomorphism. Here $\mathrm{K}_1^{\mathrm{C}}(E)$ is the classical K_1-group of the ring E; see (A.1). Its inverse,

$$\theta_E^{-1} = \det{}_E \colon \mathrm{K}_1^{\mathrm{C}}(E) \longrightarrow E^*_{\mathrm{ab}} = \mathrm{Aut}_R(M)_{\mathrm{ab}}\ ,$$

is called the *generalized determinant map*. The details are discussed in Section 4. We are now in a position to define the subgroup Ξ of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ that appears in our main Theorem (2.11) below.

(2.9) **Definition.** Let $(R, \mathfrak{m}, k)$ be a ring satisfying the hypotheses in Setup (2.1). Assume, in addition, that R is an algebra over k and that one has $\mathrm{char}(k) \neq 2$. Define a subgroup Ξ of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ as follows.

- Choose for each $1 \leqslant j \leqslant t$ and each $\alpha \in \mathrm{Aut}_R(M_j)$ elements $\beta_{j,\alpha} \in \mathrm{Aut}_R(X_j)$ and $\gamma_{j,\alpha} \in \mathrm{Aut}_R(\tau(M_j))$ as in Construction (2.8).
- Let $\tilde{\alpha}$, $\tilde{\beta}_{j,\alpha}$, and $\tilde{\gamma}_{j,\alpha}$ be the invertible matrices with entries in E obtained by applying the tilde construction (2.5) to α, $\beta_{j,\alpha}$, and $\gamma_{j,\alpha}$.

Define Ξ to be the subgroup of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ generated by the elements

$$(\det{}_E \tilde{\alpha})(\det{}_E \tilde{\beta}_{j,\alpha})^{-1}(\det{}_E \tilde{\gamma}_{j,\alpha})\ ,$$

where j ranges over $\{1, \ldots, t\}$ and α over $\mathrm{Aut}_R(M_j)$.

(2.10) **Remark.** In specific examples it is convenient to consider the simplest possible representation generator $M = M_0 \oplus M_1 \oplus \cdots \oplus M_t$. In this case, Example (2.6) shows that $\tilde{\alpha}$ and $\tilde{\gamma}_{j,\alpha}$ are 1×1 matrices with entries in E, that is, $\tilde{\alpha}, \tilde{\gamma}_{j,\alpha} \in E^*$, and consequently $\det_E \tilde{\alpha} = \tilde{\alpha}$ and $\det_E \tilde{\gamma}_{j,\alpha} = \tilde{\gamma}_{j,\alpha}$ as elements in E^*_{ab}.

We are now in a position to state our main result.

(2.11) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a ring satisfying the hypotheses in Setup (2.1). Assume that R is an algebra over its residue field k with $\mathrm{char}(k) \neq 2$, and that the Auslander–Reiten homomorphism $\Upsilon \colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ from Definition (2.2) is injective.*

Let M be any representation generator of $\mathsf{MCM}\,R$. There is an isomorphism,

$$\mathrm{K}_1(\mathsf{mod}\,R) \cong \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi\ ,$$

where Ξ is the subgroup of $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ given in Definition (2.9).

Furthermore, if $\mathrm{inc}\colon \mathsf{proj}\,R \to \mathsf{mod}\,R$ *is the inclusion functor and* $M = R \oplus M'$, *then* $\mathrm{K}_1(\mathrm{inc})\colon \mathrm{K}_1(\mathsf{proj}\,R) \to \mathrm{K}_1(\mathsf{mod}\,R)$ *may be identified with the homomorphism,*

$$\lambda\colon R^* \longrightarrow \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi \qquad \textit{given by} \qquad r \longmapsto \begin{pmatrix} r1_R & 0 \\ 0 & 1_{M'} \end{pmatrix}.$$

As mentioned in the Introduction, the proof of Theorem (2.11) spans Sections 3 to 7. Applications and examples are presented in Sections 8 and 9.

3. Coherent Pairs

We recall some results and notions from the paper [3] by Auslander and Reiten which are central in the proof of our main Theorem (2.11). Throughout this section, $\mathcal{A}$ denotes a skeletally small additive category.

(3.1) **Definition.** A *pseudo* (or *weak*) kernel of a morphism $g\colon A \to A'$ in $\mathcal{A}$ is a morphism $f\colon A'' \to A$ in $\mathcal{A}$ such that $gf = 0$, and which satisfies that every diagram in $\mathcal{A}$ as below can be completed (but not necessarily in a unique way).

$$\begin{tikzcd} & B \arrow[dl, dashed] \arrow[d, "h"] \arrow[dr, "0"] & \\ A'' \arrow[r, "f"'] & A \arrow[r, "g"'] & A'. \end{tikzcd}$$

We say that $\mathcal{A}$ *has pseudo kernels* if every morphism in $\mathcal{A}$ has a pseudo kernel.

(3.2) **Observation.** Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathcal{M}$. If $\mathcal{A}$ is precovering (or contravariantly finite) in $\mathcal{M}$, see [8, def. 5.1.1], then $\mathcal{A}$ has pseudo kernels. Indeed, if $i\colon K \to A$ is the kernel in $\mathcal{M}$ of $g\colon A \to A'$ in $\mathcal{A}$, and if $f\colon A'' \to K$ is an $\mathcal{A}$-precover of K, then $if\colon A'' \to A$ is a pseudo kernel of g.

(3.3) **Definition.** Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. Auslander and Reiten [3] call $(\mathcal{A}, \mathcal{B})$ a *coherent pair* if $\mathcal{A}$ has pseudo kernels in the sense of Definition (3.1), and $\mathcal{B}$ is precovering in $\mathcal{A}$.

If $(\mathcal{A}, \mathcal{B})$ is a coherent pair then also $\mathcal{B}$ has pseudo kernels by [3, prop. 1.4(a)].

(3.4) **Definition.** Write $\mathsf{Mod}\,\mathcal{A}$ for the abelian category of additive contravariant functors $\mathcal{A} \to \mathsf{Ab}$, where Ab is the category of abelian groups. Denote by $\mathsf{mod}\,\mathcal{A}$ the full subcategory of $\mathsf{Mod}\,\mathcal{A}$ consisting of finitely presented functors.

(3.5) If the category $\mathcal{A}$ has pseudo kernels then $\mathsf{mod}\,\mathcal{A}$ is abelian, and the inclusion functor $\mathsf{mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}$ is exact, see [3, prop. 1.3].

If $(\mathcal{A}, \mathcal{B})$ is a coherent pair, see (3.3), then the exact restriction $\mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{B}$ maps $\mathsf{mod}\,\mathcal{A}$ to $\mathsf{mod}\,\mathcal{B}$ by [3, prop. 1.4(b)]. In this case, there are functors,

$$\mathrm{Ker}\,r \xrightarrow{\;i\;} \mathsf{mod}\,\mathcal{A} \xrightarrow{\;r\;} \mathsf{mod}\,\mathcal{B}\,, \tag{3.5.1}$$

where r is the restriction and i the inclusion functor. The kernel of r, that is,

$$\mathrm{Ker}\,r = \{F \in \mathsf{mod}\,\mathcal{A} \mid F(B) = 0 \text{ for all } B \in \mathcal{B}\}\,,$$

is a Serre subcategory of the abelian category $\mathsf{mod}\,\mathcal{A}$. The quotient $(\mathsf{mod}\,\mathcal{A})/(\mathrm{Ker}\,r)$, in the sense of Gabriel [10], is equivalent to the category $\mathsf{mod}\,\mathcal{B}$, and the canonical functor $\mathsf{mod}\,\mathcal{A} \to (\mathsf{mod}\,\mathcal{A})/(\mathrm{Ker}\,r)$ may be identified with r. These assertions are

proved in [3, prop. 1.5]. Therefore (3.5.1) induces by Quillen's localization theorem [15, §5 thm. 5] a long exact sequence of K-groups,

$$\begin{aligned} \cdots \longrightarrow \mathrm{K}_n(\operatorname{Ker} r) &\xrightarrow{\mathrm{K}_n(i)} \mathrm{K}_n(\mathsf{mod}\,\mathcal{A}) \xrightarrow{\mathrm{K}_n(r)} \mathrm{K}_n(\mathsf{mod}\,\mathcal{B}) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{K}_0(\operatorname{Ker} r) &\xrightarrow{\mathrm{K}_0(i)} \mathrm{K}_0(\mathsf{mod}\,\mathcal{A}) \xrightarrow{\mathrm{K}_0(r)} \mathrm{K}_0(\mathsf{mod}\,\mathcal{B}) \longrightarrow 0\,. \end{aligned} \tag{3.5.2}$$

4. Semilocal Rings

A ring A is semilocal if $A/\mathrm{J}(A)$ is semisimple. Here $\mathrm{J}(A)$ is the Jacobson radical of A. If A is commutative then this definition is equivalent to A having only finitely many maximal ideals; see Lam [12, prop. (20.2)].

(4.1) **Lemma.** *Let R be a commutative noetherian semilocal ring, and let $M \neq 0$ be a finitely generated R-module. Then the ring $\operatorname{End}_R(M)$ is semilocal.*

Proof. As R is commutative and noetherian, $\operatorname{End}_R(M)$ is a module-finite R-algebra. Since R is semilocal, the assertion now follows from [12, prop. (20.6)]. ☐

(4.2) Denote by A^* the group of units in a ring A, and let $\vartheta_A \colon A^* \to \mathrm{K}_1^{\mathrm{C}}(A)$ be the composite of the group homomorphisms,

$$A^* \cong \mathrm{GL}_1(A) \hookrightarrow \mathrm{GL}(A) \twoheadrightarrow \mathrm{GL}(A)_{\mathrm{ab}} = \mathrm{K}_1^{\mathrm{C}}(A)\ . \tag{4.2.1}$$

Some authors refer to ϑ_A as the Whitehead determinant. If A is semilocal, then ϑ_A is surjective by Bass [6, V§9 thm. (9.1)]. As the group $\mathrm{K}_1^{\mathrm{C}}(A)$ is abelian one has $[A^*,A^*] \subseteq \operatorname{Ker}\vartheta_A$, and we write $\theta_A \colon A^*_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{C}}(A)$ for the induced homomorphism.

Vaserstein [19] showed that the inclusion $[A^*,A^*] \subseteq \operatorname{Ker}\vartheta_A$ is strict for the semilocal ring $A = \mathrm{M}_2(\mathbb{F}_2)$ where $\mathbb{F}_2$ is the field with two elements. In [19, thm. 3.6(a)] it is shown that if A is semilocal, then $\operatorname{Ker}\vartheta_A$ is the subgroup of A^* generated by elements of the form $(1+ab)(1+ba)^{-1}$ where $a, b \in A$ and $1+ab \in A^*$.

If A is semilocal, that is, $A/\mathrm{J}(A)$ is semisimple, then by the Artin–Wedderburn theorem there is an isomorphism of rings,

$$A/\mathrm{J}(A) \cong \mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_t}(D_t)\ ,$$

where $D_1,\dots,D_t$ are division rings, and $n_1,\dots,n_t$ are natural numbers all of which are uniquely determined by A. The next result is due to Vaserstein [20, thm. 2].

(4.3) **Theorem.** *Let A be semilocal and write $A/\mathrm{J}(A) \cong \mathrm{M}_{n_1}(D_1)\times\cdots\times\mathrm{M}_{n_t}(D_t)$. If none of the $\mathrm{M}_{n_i}(D_i)$'s is $\mathrm{M}_2(\mathbb{F}_2)$, and at most one of the $\mathrm{M}_{n_i}(D_i)$'s is $\mathrm{M}_1(\mathbb{F}_2) = \mathbb{F}_2$ then one has $\operatorname{Ker}\vartheta_A = [A^*,A^*]$. In particular, ϑ_A induces an isomorphism,*

$$\theta_A \colon A^*_{\mathrm{ab}} \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{C}}(A)\ .$$ ☐

(4.4) **Remark.** Note that if A is a semilocal ring which is an algebra over a field k with characteristic $\neq 2$, then the hypothesis in Theorem (4.3) is satisfied.

If A is a commutative semilocal ring, then $\operatorname{Ker}\vartheta_A$ and the commutator subgroup $[A^*,A^*] = \{1\}$ are identical, i.e. the surjective homomorphism $\vartheta_A = \theta_A \colon A^* \to \mathrm{K}_1^{\mathrm{C}}(A)$ is an isomorphism. Indeed, the determinant homomorphisms $\det_n \colon \mathrm{GL}_n(A) \to A^*$ induce a homomorphism $\det_A \colon \mathrm{K}_1^{\mathrm{C}}(A) \to A^*$ that evidently satisfies $\det_A \theta_A = 1_{A^*}$. Since θ_A is surjective, it follows that θ_A is an isomorphism with $\theta_A^{-1} = \det_A$.

(4.5) **Definition.** Let A be a ring for which the homomorphism $\theta_A \colon A^*_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{C}}(A)$ from (4.2) is an isomorphism; for example, A could be a commutative semilocal ring or a noncommutative semilocal ring satisfying the assumptions in Theorem (4.3). The inverse θ_A^{-1} is denoted by $\det_A$, and we call it the *generalized determinant.*

(4.6) **Remark.** Let ξ be an $m \times n$ and let χ be an $n \times p$ matrix with entries in a ring A. Denote by "$\cdot$" the product $\mathrm{M}_{m\times n}(A^{\mathrm{op}}) \times \mathrm{M}_{n\times p}(A^{\mathrm{op}}) \to \mathrm{M}_{m\times p}(A^{\mathrm{op}})$. Then

$$(\xi \cdot \chi)^T = \chi^T \xi^T \ ,$$

where $\chi^T\xi^T$ is computed using the product $\mathrm{M}_{p\times n}(A) \times \mathrm{M}_{n\times m}(A) \to \mathrm{M}_{p\times m}(A)$. Thus, transposition $(-)^T \colon \mathrm{GL}_n(A^{\mathrm{op}}) \to \mathrm{GL}_n(A)$ is an anti-isomorphism (this is also noted in [6, V§7]), which induces an isomorphism $(-)^T \colon \mathrm{K}_1^{\mathrm{C}}(A^{\mathrm{op}}) \to \mathrm{K}_1^{\mathrm{C}}(A)$.

(4.7) **Lemma.** *Let A be a ring for which the generalized determinant $\det_A = \theta_A^{-1}$ exists; cf. Definition (4.5). For every invertible matrix ξ with entries in A one has an equality $\det_{A^{\mathrm{op}}}(\xi^T) = \det_A(\xi)$ in the abelian group $(A^{\mathrm{op}})^*_{\mathrm{ab}} = A^*_{\mathrm{ab}}$.*

Proof. Clearly, there is a commutative diagram,

$$\begin{tikzcd} A^*_{\mathrm{ab}} \arrow[r, equal] \arrow[d, "\theta_A"', "\cong"] & (A^{\mathrm{op}})^*_{\mathrm{ab}} \arrow[d, "\cong"', "\theta_{A^{\mathrm{op}}}"] \\ \mathrm{K}_1^{\mathrm{C}}(A) \arrow[r, "\cong", "(-)^T"'] & \mathrm{K}_1^{\mathrm{C}}(A^{\mathrm{op}}) \, , \end{tikzcd}$$

It follows that one has $\theta_{A^{\mathrm{op}}}^{-1} \circ (-)^T = \theta_A^{-1}$, that is, $\det_{A^{\mathrm{op}}} \circ (-)^T = \det_A$. $\square$

5. Some Useful Functors

Throughout this section, A is a ring and M is a fixed left A-module. We denote by $E = \mathrm{End}_A(M)$ the endomorphism ring of M. Note that $M = {}_{A,E}M$ has a natural left-A-left-E–bimodule structure.

(5.1) There is a pair of adjoint functors,

$$\mathsf{Mod}\, A \underset{-\otimes_E M}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \mathsf{Mod}(E^{\mathrm{op}}) \ .$$

It is easily seen that they restrict to a pair of quasi-inverse equivalences,

$$\mathsf{add}_A M \underset{-\otimes_E M}{\overset{\mathrm{Hom}_A(M,-)}{\underset{\simeq}{\rightleftarrows}}} \mathsf{proj}(E^{\mathrm{op}}) \ .$$

Auslander referred to this phenomenon as *projectivization*; see [5, I§2].

Let $F \in \mathsf{Mod}(\mathsf{add}_A M)$, that is, $F \colon \mathsf{add}_A M \to \mathsf{Ab}$ is a contravariant additive functor, see Definition (3.4). The compatible E-module structure on the given A-module M induces an E^{op}-module structure on the abelian group FM which is given by $z\alpha = (F\alpha)(z)$ for $\alpha \in E$ and $z \in FM$.

(5.2) **Proposition.** *There are quasi-inverse equivalences of abelian categories,*

$$\mathsf{Mod}(\mathsf{add}_A M) \underset{f_M}{\overset{e_M}{\underset{\simeq}{\rightleftarrows}}} \mathsf{Mod}(E^{\mathrm{op}}) \ ,$$

where e_M (evaluation) and f_M (functorfication) are defined as follows,

$$e_M(F) = FM \qquad \text{and} \qquad f_M(Z) = Z \otimes_E \mathrm{Hom}_A(-,M)|_{\mathsf{add}_A M}\ ,$$

for F in $\mathsf{Mod}(\mathsf{add}_A M)$ and Z in $\mathsf{Mod}(E^{\mathrm{op}})$. They restrict to quasi-inverse equivalences between categories of finitely presented objects,

$$\mathsf{mod}(\mathsf{add}_A M) \underset{f_M}{\overset{e_M}{\rightleftarrows}} \mathsf{mod}(E^{\mathrm{op}})\ .$$

Proof. For Z in $\mathsf{Mod}(E^{\mathrm{op}})$ the canonical isomorphism

$$Z \xrightarrow{\cong} Z \otimes_E E = Z \otimes_E \mathrm{Hom}_A(M,M) = e_M f_M(Z)$$

is natural in Z. Thus, the functors $\mathrm{id}_{\mathsf{Mod}(E^{\mathrm{op}})}$ and $e_M f_M$ are naturally isomorphic. For F in $\mathsf{Mod}(\mathsf{add}_A M)$ there is a natural transformation,

$$f_M e_M(F) = FM \otimes_E \mathrm{Hom}_A(-,M)|_{\mathsf{add}_A M} \xrightarrow{\delta} F\ ; \tag{5.2.1}$$

for X in $\mathsf{add}_A M$ the homomorphism $\delta_X \colon FM \otimes_E \mathrm{Hom}_A(X,M) \to FX$ is given by $z \otimes \psi \mapsto (F\psi)(z)$. Note that δ_M is an isomorphism as it may be identified with the canonical isomorphism $FM \otimes_E {}_E E \xrightarrow{\cong} FM$ in Ab. As the functors in (5.2.1) are additive, it follows that δ_X is an isomorphism for every $X \in \mathsf{add}_A M$, that is, δ is a natural isomorphism. Since (5.2.1) is natural in F, the functors $f_M e_M$ and $\mathrm{id}_{\mathsf{Mod}(\mathsf{add}_A M)}$ are naturally isomorphic.

It is straightforward to verify that the functors e_M and f_M map finitely presented objects to finitely presented objects. $\square$

(5.3) **Observation.** In the case $M = A$ one has $E = \mathrm{End}_A(M) = A^{\mathrm{op}}$, and therefore Proposition (5.2) yields an equivalence $f_A \colon \mathsf{mod}\, A \to \mathsf{mod}(\mathsf{proj}\, A)$ given by

$$X \longmapsto X \otimes_{A^{\mathrm{op}}} \mathrm{Hom}_A(-,A)|_{\mathsf{proj}\, A}\ .$$

It is easily seen that the functor f_A is naturally isomorphic to the functor given by

$$X \longmapsto \mathrm{Hom}_A(-,X)|_{\mathsf{proj}\, A}\ .$$

We will usually identify f_A with this functor.

(5.4) **Definition.** The functor $y_M \colon \mathsf{add}_A M \to \mathsf{mod}(\mathsf{add}_A M)$ which for $X \in \mathsf{add}_A M$ is given by $y_M(X) = \mathrm{Hom}_A(-,X)|_{\mathsf{add}_A M}$ is called the *Yoneda functor*.

Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathcal{M}$. If $\mathcal{A}$ is closed under extensions in $\mathcal{M}$ then $\mathcal{A}$ has a natural exact structure. However, one can always equip $\mathcal{A}$ with the *trivial exact structure*. In this structure, the "exact sequences" (somtimes called *conflations*) are only the split exact ones. When viewing $\mathcal{A}$ as an exact category with the trivial exact structure, we denote it by $\mathcal{A}_0$.

(5.5) **Lemma.** *Assume that A is commutative and noetherian and let $M \in \mathsf{mod}\, A$. Set $E = \mathrm{End}_A(M)$ and assume that E^{op} has finite global dimension. For the exact Yoneda functor $y_M \colon (\mathsf{add}_A M)_0 \to \mathsf{mod}(\mathsf{add}_A M)$, see (5.4), the homomorphisms $\mathrm{K}_n(y_M)$, where $n \geqslant 0$, and $\mathrm{K}_1^{\mathrm{B}}(y_M)$ are isomorphisms.*

Proof. By application of K_n to the commutative diagram,

$$\begin{tikzcd} (\mathsf{add}_A M)_0 \arrow[r, "\mathrm{Hom}_A(M,-)", "\simeq"'] \arrow[d, "y_M"'] & \mathsf{proj}(E^{\mathrm{op}}) \arrow[d, "\mathrm{inc}"] \\ \mathsf{mod}(\mathsf{add}_A M) \arrow[r, "\simeq", "e_M"'] & \mathsf{mod}(E^{\mathrm{op}}) \end{tikzcd} ,$$

it follows that $\mathrm{K}_n(y_M)$ is an isomorphism if and only if $\mathrm{K}_n(\mathrm{inc})$ is an isomorphism. The latter holds by Quillen's resolution theorem [15, §4 thm. 3], since E^{op} has finite global dimension. A similar argument shows that $\mathrm{K}_1^{\mathrm{B}}(y_M)$ is an isomorphism. This time one needs to apply Bass' resolution theorem; see [6, VIII§4 thm. (4.6)]. □

Since K_0 may be identified with the Grothendieck group functor, cf. (A.6), the following result is well-known. In any case, it is straightforward to verify.

(5.6) **Lemma.** *Assume that* $\mathsf{mod}\,A$ *is Krull–Schmidt. Let* $N = N_1^{n_1} \oplus \cdots \oplus N_s^{n_s}$ *be a finitely generated A-module, where $N_1, \ldots, N_s$ are non-isomorphic indecomposable A-modules and $n_1, \ldots, n_s > 0$. The homomorphism of abelian groups,*

$$\psi_N \colon \mathbb{Z}N_1 \oplus \cdots \oplus \mathbb{Z}N_s \longrightarrow \mathrm{K}_0((\mathsf{add}_A N)_0) ,$$

given by $N_j \mapsto [N_j]$, is an isomorphism. □

6. The Abelian Category $\mathcal{Y}$

By the assumptions in Setup (2.1), the ground ring R has a dualizing module. It follows from Auslander and Buchweitz [2, thm. A] that $\mathsf{MCM}\,R$ is precovering in $\mathsf{mod}\,R$. Actually, in our case $\mathsf{MCM}\,R$ equals $\mathsf{add}_R M$ for some finitely generated R-module M (a representation generator), and it is easily seen that every category of this form is precovering in $\mathsf{mod}\,R$. By Observation (3.2) we have a coherent pair $(\mathsf{MCM}\,R, \mathsf{proj}\,R)$, which by (3.5) yields a Gabriel localization sequence,

$$\mathcal{Y} = \mathrm{Ker}\, r \xrightarrow{\;i\;} \mathsf{mod}(\mathsf{MCM}\,R) \xrightarrow{\;r\;} \mathsf{mod}(\mathsf{proj}\,R) \; . \tag{6.0.1}$$

Here r is the restriction functor, $\mathcal{Y} = \mathrm{Ker}\, r$, and i is the inclusion. Since an additive functor vanishes on $\mathsf{proj}\,R$ if and only if it vanishes on R, one has

$$\mathcal{Y} = \{ F \in \mathsf{mod}(\mathsf{MCM}\,R) \,|\, F(R) = 0 \} \; .$$

The following two results about the abelian category $\mathcal{Y}$ are due to Yoshino. The first result is [21, (13.7.4)]; the second is (proofs of) [21, lem. (4.12) and prop. (4.13)].

(6.1) **Theorem.** *Every object in the abelian category $\mathcal{Y}$ has finite length.* □

(6.2) **Theorem.** *Consider for $1 \leqslant j \leqslant t$ the Auslander–Reiten sequence (2.1.1) ending in M_j. The functor F_j, defined by the following exact sequence in* $\mathsf{mod}(\mathsf{MCM}\,R)$,

$$0 \longrightarrow \mathrm{Hom}_R(-, \tau(M_j)) \longrightarrow \mathrm{Hom}_R(-, X_j) \longrightarrow \mathrm{Hom}_R(-, M_j) \longrightarrow F_j \longrightarrow 0 \; ,$$

is a simple object in $\mathcal{Y}$. Conversely, every simple functor in $\mathcal{Y}$ is naturally isomorphic to F_j for some $1 \leqslant j \leqslant t$. □

(6.3) **Proposition.** *Let $i \colon \mathcal{Y} \to$* $\mathsf{mod}(\mathsf{MCM}\,R)$ *be the inclusion functor from (6.0.1) and let $\Upsilon \colon \mathbb{Z}^t \to \mathbb{Z}^{t+1}$ be the Auslander–Reiten homomorphism; see Definition (2.2). The homomorphisms $\mathrm{K}_0(i)$ and Υ are isomorphic.*

Proof. We claim that the following diagram of abelian groups is commutative,

$$\begin{tikzcd}
\mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t \arrow[r, "\Upsilon"] \arrow[dd, "\varphi"', "\cong"] & \mathbb{Z}M_0 \oplus \mathbb{Z}M_1 \oplus \cdots \oplus \mathbb{Z}M_t \arrow[d, "\cong"', "\psi_M"] \\
& \mathrm{K}_0((\mathsf{MCM}\,R)_0) \arrow[d, "\cong"', "\mathrm{K}_0(y_M)"] \\
\mathrm{K}_0(\mathcal{Y}) \arrow[r, "\mathrm{K}_0(i)"] & \mathrm{K}_0(\mathsf{mod}(\mathsf{MCM}\,R))\ .
\end{tikzcd}$$

The homomorphism φ is defined by $M_j \mapsto [F_j]$ where $F_j \in \mathcal{Y}$ is described in (6.2). From Theorems (6.1) and (6.2) and the proof of Rosenberg [16, thm. 3.1.8(1)], it follows that φ is an isomorphism. The module M is a representation generator for $\mathsf{MCM}\,R$, see (2.1), and ψ_M is the isomorphism given in Lemma (5.6). Finally, y_M is the Yoneda functor from Definition (5.4). By Leuschke [13, thm. 6] the ring E^{op}, where $E = \mathrm{End}_R(M)$, has finite global dimension, and thus Lemma (5.5) implies that $\mathrm{K}_0(y_M)$ is an isomorphism.

To prove that the diagram is commutative, note that $(\mathrm{K}_0(i)\varphi)(M_j) = [F_j]$. Furthermore one has $\Upsilon(M_j) = y_{0j}M_0 + \cdots + y_{tj}M_t$ where $(y_{0j}, \ldots, y_{tj})$ is the j'th column in the Auslander–Reiten $(t+1) \times t$ matrix Υ. By Definition (2.2) one has

$$\Upsilon(M_j) = \tau(M_j) + M_j - n_{0j}M_0 - \cdots - n_{tj}M_t\ ,$$

where $0 \to \tau(M_j) \to X_j \to M_j \to 0$ is the j'th Auslander–Reiten sequence (2.1.1) and $X_j \cong M_0^{n_{0j}} \oplus \cdots \oplus M_t^{n_{tj}}$. Thus, the definitions of ψ_M and $\mathrm{K}_0(y_M)$ yield

$$\begin{aligned}
(\mathrm{K}_0(y_M)\psi_M\Upsilon)(M_j) &= [(-,\tau(M_j)) + (-,M_j) - n_{0j}(-,M_0) - \cdots - n_{tj}(-,M_t)] \\
&= [(-,\tau(M_j)) + (-,M_j) - (-,X_j)] = [F_j]\ ,
\end{aligned}$$

where the last equality is by the exact sequence in Theorem (6.2). In the display above, we have used the shorthand notation $(-,Z)$ for $\mathrm{Hom}_R(-,Z)|_{\mathsf{MCM}\,R}$. ☐

7. Proof of the Main Theorem

Throughout, we fix the assumptions and the notation in Setup (2.1). We begin with a result on the Gersten–Sherman transformation from Appendix A.

(7.1) **Lemma.** *$\zeta_{\mathcal{C}} \colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}) \to \mathrm{K}_1(\mathcal{C})$ is an isomorphism for $\mathcal{C} = \mathsf{mod}(\mathsf{MCM}\,R)$.*

Proof. As ζ is a natural transformation, there is a commutative diagram,

$$\begin{tikzcd}
\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}})) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})"] \arrow[d, "\zeta_{\mathsf{proj}(E^{\mathrm{op}})}"'] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}})) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(f_M)"] \arrow[d, "\zeta_{\mathsf{mod}(E^{\mathrm{op}})}"] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) \arrow[d, "\zeta_{\mathsf{mod}(\mathsf{MCM}\,R)}"] \\
\mathrm{K}_1(\mathsf{proj}(E^{\mathrm{op}})) \arrow[r, "\mathrm{K}_1(\mathrm{inc})"] & \mathrm{K}_1(\mathsf{mod}(E^{\mathrm{op}})) \arrow[r, "\mathrm{K}_1(f_M)"] & \mathrm{K}_1(\mathsf{mod}(\mathsf{MCM}\,R))\ ,
\end{tikzcd}$$

where $f_M \colon \mathsf{mod}(E^{\mathrm{op}}) \to \mathsf{mod}(\mathsf{MCM}\,R)$ is the equivalence from Proposition (5.2) and inc is the inclusion of $\mathsf{proj}(E^{\mathrm{op}})$ into $\mathsf{mod}(E^{\mathrm{op}})$.

From Leuschke [13, thm. 6], the noetherian ring E^{op} has finite global dimension. Hence Bass' and Quillen's resolution theorems, [6, VIII§4 thm. (4.6)] (see also Rosenberg [16, thm. 3.1.14]) and [15, §4 thm. 3], imply that $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$ and $\mathrm{K}_1(\mathrm{inc})$ are isomorphisms. Since f_M is an equivalence, $\mathrm{K}_1^{\mathrm{B}}(f_M)$ and $\mathrm{K}_1(f_M)$ are isomorphisms as well. Consequently, $\zeta_{\mathsf{mod}(\mathsf{MCM}\,R)}$ is an isomorphism if and only if $\zeta_{\mathsf{proj}(E^{\mathrm{op}})}$ is an isomorphism, and the latter holds by Theorem (A.7). ☐

We proceed with a result on Bass' K_1-functor.

(7.2) **Remark.** Let $\mathcal{C}$ be an exact category. As in the paragraph preceding Lemma (5.5), we denote by $\mathcal{C}_0$ the category $\mathcal{C}$ equipped with the trivial exact structure. Note that the identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C}_0 \to \mathcal{C}$ is exact and the induced homomorphism $\mathrm{K}_1^{\mathrm{B}}(\mathrm{id}_{\mathcal{C}})\colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}_0) \to \mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$ is surjective, indeed, one has $\mathrm{K}_1^{\mathrm{B}}(\mathrm{id}_{\mathcal{C}})\big([C,\alpha]\big) = [C,\alpha]$.

(7.3) **Lemma.** *Consider the restriction functor* $r\colon \mathsf{mod}(\mathsf{MCM}\,R) \to \mathsf{mod}(\mathsf{proj}\,R)$ *and identity functor* $\mathrm{id}_{\mathsf{MCM}\,R}\colon (\mathsf{MCM}\,R)_0 \to \mathsf{MCM}\,R$. *The homomorphisms* $\mathrm{K}_1^{\mathrm{B}}(r)$ *and* $\mathrm{K}_1^{\mathrm{B}}(\mathrm{id}_{\mathsf{MCM}\,R})$ *are isomorphic, in particular,* $\mathrm{K}_1^{\mathrm{B}}(r)$ *is surjective by Remark (7.2).*

Proof. Consider the commutative diagram of exact categories and exact functors,

$$\begin{tikzcd}
(\mathsf{MCM}\,R)_0 \arrow[r, "\mathrm{id}_{\mathsf{MCM}\,R}"] \arrow[dd, "y_M"'] & \mathsf{MCM}\,R \arrow[d, "j"] \\
 & \mathsf{mod}\,R \arrow[d, "f_R", "\simeq"'] \\
\mathsf{mod}(\mathsf{MCM}\,R) \arrow[r, "r"] & \mathsf{mod}(\mathsf{proj}\,R)\,,
\end{tikzcd}$$

where y_M is the Yoneda functor from Definition (5.4), j is the inclusion, and f_R is the equivalence from Observation (5.3). We will prove the lemma by arguing that the vertical functors induce isomorphisms on the level of $\mathrm{K}_1^{\mathrm{B}}$.

The ring E^{op} has finite global dimension by Leuschke [13, thm. 6], and hence Lemma (5.5) gives that that $\mathrm{K}_1^{\mathrm{B}}(y_M)$ is an isomorphism. Since f_R is an equivalence, $\mathrm{K}_1^{\mathrm{B}}(f_R)$ is obviously an isomorphism. To argue that $\mathrm{K}_1^{\mathrm{B}}(j)$ is an isomorphism, we apply Bass' resolution theorem [16, thm. 3.1.14]. We must check that the subcategory $\mathsf{MCM}\,R$ of $\mathsf{mod}\,R$ satisfies conditions (1)–(3) in *loc. cit.* Condition (1) follows as $\mathsf{MCM}\,R$ is precovering in $\mathsf{mod}\,R$. As R is Cohen–Macaulay, every module in $\mathsf{mod}\,R$ has a resolution of finite length by modules in $\mathsf{MCM}\,R$, see [21, prop. (1.4)]; thus condition (2) holds. Condition (3) requires that $\mathsf{MCM}\,R$ is closed under kernels of epimorphisms; this is well-known from e.g. [21, prop. (1.3)]. □

We will also need the following classical notion.

(7.4) **Definition.** Let $\mathcal{M}$ be an abelian category, and let M be an object in $\mathcal{M}$. A *projective cover* of M is an epimorphism $\varepsilon\colon P \twoheadrightarrow M$ in $\mathcal{M}$, where P is projective, such that every endomorphism $\alpha\colon P \to P$ satisfying $\varepsilon\alpha = \varepsilon$ is an automorphism.

(7.5) **Lemma.** *Let there be given a commutative diagram,*

$$\begin{tikzcd}
P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\alpha"'] & M \arrow[d, "\varphi"] \\
P \arrow[r, two heads, "\varepsilon"] & M
\end{tikzcd}$$

in an abelian category $\mathcal{M}$*, where* $\varepsilon\colon P \twoheadrightarrow M$ *is a projective cover of* M*. If* φ *is an automorphism then* α *is an automorphism.*

Proof. As P is projective and ε is an epimorphism, there exists $\beta\colon P \to P$ such that $\varepsilon\beta = \varphi^{-1}\varepsilon$. By assumption one has $\varepsilon\alpha = \varphi\varepsilon$. Hence $\varepsilon\alpha\beta = \varphi\varepsilon\beta = \varphi\varphi^{-1}\varepsilon = \varepsilon$, and similarly, $\varepsilon\beta\alpha = \varepsilon$. As ε is a projective cover, we conclude that $\alpha\beta$ and $\beta\alpha$ are automorphisms of P, and thus α must be an automorphism. □

The following lemma explains the point of the tilde construction (2.5).

(7.6) **Lemma.** *Consider the isomorphism* $\eta_{E^{\mathrm{op}}}\colon \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}}) \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$ *given in (A.5). For* $X \in \mathsf{MCM}\,R$ *and* $\alpha \in \mathrm{Aut}_R(X)$ *consider the element*

$$\xi_{X,\alpha} = \big[\,\mathrm{Hom}_R(M,X), \mathrm{Hom}_R(M,\alpha)\,\big] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))\ .$$

Let $\tilde{\alpha}$ *be the invertible matrix with entries in* E *obtained by applying Construction (2.5) to* α*. Then* $\eta_{E^{\mathrm{op}}}^{-1}(\xi_{X,\alpha}) \in \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}})$ *is represented by the matrix* $\tilde{\alpha}^T$*.*

Proof. Write $(M,-)$ for $\mathrm{Hom}_R(M,-)$, and let $\psi\colon X \oplus Y \xrightarrow{\cong} M^q$ be as in Construction (2.5). The R-module isomorphism ψ induces an isomorphism of E^{op}-modules,

$$(M,X) \oplus (M,Y) = (M, X\oplus Y) \xrightarrow[\cong]{(M,\psi)} (M,M^q) \cong E^q\ .$$

Consider the automorphism of the free E^{op}-module E^q given by

$$(M,\psi)\big((M,\alpha)\oplus 1_{(M,Y)}\big)(M,\psi)^{-1} = (M, \psi(\alpha\oplus 1_Y)\psi^{-1}) = (M,\tilde{\alpha})\ .$$

We view elements in the R-module M^q as columns and elements in E^q as rows. The isomorphism $E^q \cong (M,M^q)$ identifies a row vector $\beta = (\beta_1,\dots,\beta_q) \in E^q$ with the R-linear map $\beta^T\colon M \to M^q$ whose coordinate functions are $\beta_1,\dots,\beta_q$. The coordinate functions of $(M,\tilde{\alpha})(\beta^T) = \tilde{\alpha}\circ\beta^T$ are the entries in the column $\tilde{\alpha}\beta^T$, where the matrix product used is $\mathrm{M}_{q\times q}(E)\times \mathrm{M}_{q\times 1}(E) \to \mathrm{M}_{q\times 1}(E)$. Thus, the action of $(M,\tilde{\alpha})$ on a row $\beta \in E^q$ is the row $(\tilde{\alpha}\beta^T)^T \in E^q$. In view of Remark (4.6) one has $(\tilde{\alpha}\beta^T)^T = \beta\cdot\tilde{\alpha}^T$, where "$\cdot$" is the product $\mathrm{M}_{1\times q}(E^{\mathrm{op}})\times \mathrm{M}_{q\times q}(E^{\mathrm{op}}) \to \mathrm{M}_{1\times q}(E^{\mathrm{op}})$. Consequently, over the ring E^{op}, the automorphism $(M,\tilde{\alpha})$ of the E^{op}-module E^q acts on row vectors by multiplication with $\tilde{\alpha}^T$ from the right. The desired conclusion now follows from (A.5). $\square$

(7.7) **Observation.** For any commutative noetherian local ring R, there is an isomorphism $\rho_R\colon R^* \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R)$ given by the composite of

$$R^* \xrightarrow[\cong]{\theta_R} \mathrm{K}_1^{\mathrm{C}}(R) \xrightarrow[\cong]{\eta_R} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R)\ .$$

The first map is described in (4.2); it is an isomorphism by Srinivas [18, exa. (1.6)]. The second isomorphism is discussed in (A.5). Thus, ρ_R maps $r \in R^*$ to $[R, r1_R]$.

Proof of Theorem (2.11). The Gabriel localization sequence (6.0.1) induces by (3.5) a long exact sequence of Quillen K-groups,

$$\cdots \longrightarrow \mathrm{K}_1(\mathcal{Y}) \xrightarrow{\mathrm{K}_1(i)} \mathrm{K}_1(\mathsf{mod}(\mathsf{MCM}\,R)) \xrightarrow{\mathrm{K}_1(r)} \mathrm{K}_1(\mathsf{mod}(\mathsf{proj}\,R)) \longrightarrow \mathrm{K}_0(\mathcal{Y}) \xrightarrow{\mathrm{K}_0(i)} \cdots$$

By Proposition (6.3), we may identify $\mathrm{K}_0(i)$ with the Auslander–Reiten homomorphism, which is assumed to be injective. Therefore, the bottom row in the following commutative diagram of abelian groups is exact,

$$\begin{tikzcd}
\mathrm{K}_1^{\mathrm{B}}(\mathcal{Y}) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(i)"] \arrow[d, "\zeta_{\mathcal{Y}}", "\cong"'] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(r)"] \arrow[d, "\zeta_{\mathsf{mod}(\mathsf{MCM}\,R)}", "\cong"'] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{proj}\,R)) \arrow[r] \arrow[d, "\zeta_{\mathsf{mod}(\mathsf{proj}\,R)}"] & 0 \\
\mathrm{K}_1(\mathcal{Y}) \arrow[r, "\mathrm{K}_1(i)"] & \mathrm{K}_1(\mathsf{mod}(\mathsf{MCM}\,R)) \arrow[r, "\mathrm{K}_1(r)"] & \mathrm{K}_1(\mathsf{mod}(\mathsf{proj}\,R)) \arrow[r] & 0.
\end{tikzcd} \tag{7.7.1}$$

The vertical homomorphisms are given by the Gersten–Sherman transformation; see Appendix A. It follows from Theorems (6.1) and (6.2) that $\mathcal{Y}$ is a length category with only finitely many simple objects; thus $\zeta_{\mathcal{Y}}$ is an isomorphism by Theorem (A.8). And $\zeta_{\mathsf{mod}(\mathsf{MCM}\,R)}$ is an isomorphism by Lemma (7.1). As $ri = 0$ it follows that $\mathrm{K}_1^{\mathrm{B}}(r)\mathrm{K}_1^{\mathrm{B}}(i) = 0$, and a diagram chase now shows that $\operatorname{Im}\mathrm{K}_1^{\mathrm{B}}(i) = \operatorname{Ker}\mathrm{K}_1^{\mathrm{B}}(r)$. Furthermore $\mathrm{K}_1^{\mathrm{B}}(r)$ is surjective by Lemma (7.3), so the top row in (7.7.1) is exact as well. The Five Lemma now implies that $\zeta_{\mathsf{mod}(\mathsf{proj}\,R)}$ is an isomorphism. Since the category $\mathsf{mod}(\mathsf{proj}\,R)$ is equivalent to $\mathsf{mod}\,R$, see Observation (5.3), it follows that $\zeta_{\mathsf{mod}\,R}$ is an isomorphism as well. Thus, Quillen's K-group $\mathrm{K}_1(\mathsf{mod}\,R)$, which we wish to compute, can naturally be identified with Bass' K-group $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$.

By the relations that define $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$, see (A.3), there is a homomorphism $\pi_0\colon \operatorname{Aut}_R(M) \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$ given by $\alpha \mapsto [M,\alpha]$. Since $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$ is abelian, π_0 induces a homomorphism $\pi\colon \operatorname{Aut}_R(M)_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$. We claim that π fits into the following commutative diagram,

$$\begin{tikzcd}
\operatorname{Aut}_R(M)_{\mathrm{ab}} \arrow[r, "\pi"] \arrow[d, "\sigma"', "\cong"] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R) \arrow[d, "\cong"', "\mathrm{K}_1^{\mathrm{B}}(f_R)"] \\
\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R)) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(r)"] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{proj}\,R))\,.
\end{tikzcd} \tag{7.7.2}$$

Here the isomorphism σ is defined as the composite of the following isomorphisms,

$$\begin{aligned}
\operatorname{Aut}_R(M)_{\mathrm{ab}} = E^*_{\mathrm{ab}} = (E^{\mathrm{op}})^*_{\mathrm{ab}} \;&\xrightarrow[\cong]{\theta_{E^{\mathrm{op}}}}\; \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}})\\
&\xrightarrow[\cong]{\eta_{E^{\mathrm{op}}}}\; \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))\\
&\xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{B}}(\jmath)}\; \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}}))\\
&\xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{B}}(f_M)}\; \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))\,.
\end{aligned}$$

The ring E, and hence also its opposite ring E^{op}, is semilocal by Lemma (4.1). By assumption, R is a k-algebra, and hence so is E^{op}. Thus, in view of Remark (4.4) and the assumption $\operatorname{char}(k) \neq 2$, we get the isomorphism $\theta_{E^{\mathrm{op}}}$ from Theorem (4.3). It maps $\alpha \in \operatorname{Aut}_R(M)_{\mathrm{ab}}$ to the image of the 1×1 matrix $(\alpha) \in \mathrm{GL}(E^{\mathrm{op}})$ in $\mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}})$.

The isomorphism $\eta_{E^{\mathrm{op}}}$ is described in (A.5); it maps $\xi \in \mathrm{GL}_n(E^{\mathrm{op}})$ to the class $[(E_E)^n, \xi] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$.

The third homomorphism in the display above is induced by the inclusion functor $\jmath\colon \mathsf{proj}(E^{\mathrm{op}}) \to \mathsf{mod}(E^{\mathrm{op}})$. By Leuschke [13, thm. 6] the noetherian ring E^{op} has finite global dimension and hence Bass' resolution theorem [6, VIII§4 thm. (4.6)] (see also Rosenberg [16, thm. 3.1.14]) implies that $\mathrm{K}_1^{\mathrm{B}}(\jmath)$ is an isomorphism. It maps an element $[P,\alpha] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$ to $[P,\alpha] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}}))$.

The fourth and last isomorphism $\mathrm{K}_1^{\mathrm{B}}(f_M)$ in the display is induced by the equivalence $f_M\colon \mathsf{mod}(E^{\mathrm{op}}) \to \mathsf{mod}(\mathsf{MCM}\,R)$ from Proposition (5.2).

We summarize: The isomorphism $\sigma\colon \operatorname{Aut}_R(M)_{\mathrm{ab}} \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))$ maps an element $\alpha \in \operatorname{Aut}_R(M)_{\mathrm{ab}}$ to the class

$$\big[E_E \otimes_E \operatorname{Hom}_R(-,M)|_{\mathsf{MCM}\,R}\,,\,(\alpha\,\cdot) \otimes_E \operatorname{Hom}_R(-,M)|_{\mathsf{MCM}\,R}\big]\,,$$

which is evidently the same as the class

$$\big[\operatorname{Hom}_R(-,M)|_{\mathsf{MCM}\,R}\,,\,\operatorname{Hom}_R(-,\alpha)|_{\mathsf{MCM}\,R}\big]\,.$$

It is now straightforward to see that the diagram (7.7.2) is commutative, indeed, $\mathrm{K}_1^{\mathrm{B}}(r)\sigma$ and $\mathrm{K}_1^{\mathrm{B}}(f_R)\pi$ both map $\alpha \in \mathrm{Aut}_R(M)_{\mathrm{ab}}$ to the class

$$\big[\,\mathrm{Hom}_R(-,M)|_{\mathsf{proj}\,R}\,,\,\mathrm{Hom}_R(-,\alpha)|_{\mathsf{proj}\,R}\big]\ .$$

Since $\mathrm{K}_1^{\mathrm{B}}(r)$ is surjective, so is π. Exactness of the top row in (7.7.1) and commutativity of (7.7.2) show that $\mathrm{Ker}\,\pi = \sigma^{-1}(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$. Thus, if we can show the equality

$$\sigma^{-1}(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i)) = \Xi\ , \tag{7.7.3}$$

then it follows that $\pi\colon\ \mathrm{Aut}_R(M)_{\mathrm{ab}} \twoheadrightarrow \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$ induces an isomorphism,

$$\widehat{\pi}\colon\ \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)\ ,$$

which proves the first assertion in Theorem (2.11). We will shortly prove the equality (7.7.3), however, first we explain how the theorem's last assertion follows.

Note that the Gersten–Sherman transformation identifies the homomorphisms $\mathrm{K}_1(\mathrm{inc})$ and $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$; indeed $\zeta_{\mathsf{proj}\,R}$ is an isomorphism by Theorem (A.7) and it is proved above that $\zeta_{\mathsf{mod}\,R}$ is an isomorphism as well. Thus, we must show that $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$ can be identified with the homomorphism $\lambda\colon R^* \to \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi$ given by $r \mapsto r1_R \oplus 1_{M'}$. To this end, consider the isomorphism $\rho_R\colon R^* \to \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R)$ from Observation (7.7) given by $r \mapsto [R, r1_R]$. The fact that $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})$ and λ are isomorphic maps now follows the diagram,

$$\begin{tikzcd}
R^* \arrow[r, "\lambda"] \arrow[d, "\rho_R"', "\cong"] & \mathrm{Aut}_R(M)_{\mathrm{ab}}/\Xi \arrow[d, "\cong"', "\widehat{\pi}"] \\
\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})"] & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)\ ,
\end{tikzcd}$$

which is commutative. Indeed, for $r \in R^*$ one has

$$(\widehat{\pi}\lambda)(r) = [M, r1_R \oplus 1_{M'}] = [R, r1_R] + [M', 1_{M'}] = [R, r1_R] = (\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})\rho_R)(r)\ ,$$

where the penultimate equality is by (A.4).

It remains to prove the equality (7.7.3). By Theorems (6.1) and (6.2) every element in the $\mathcal{Y}$ has finite length and the simple objects in $\mathcal{Y}$ are (up to isomorphism) exactly the functors $F_1,\dots,F_t$. Thus, by (the proof of) [16, thm. 3.1.8(2)], the abelian group $\mathrm{K}_1^{\mathrm{B}}(\mathcal{Y})$ is generated by elements $[F_j,\varphi]$, where $j = 1,\dots,t$ and φ is an automorphism of F_j. It follows that:

- The subgroup $\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i)$ of $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(\mathsf{MCM}\,R))$ is generated by $[F_j,\varphi]$, where j ranges over $\{1,\dots,t\}$ and φ over all automorphisms of F_j.

By definition of σ, one has

$$\sigma^{-1} = \det{}_{E^{\mathrm{op}}}\eta_{E^{\mathrm{op}}}^{-1}\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}\mathrm{K}_1^{\mathrm{B}}(f_M)^{-1}\ ,$$

where $\det_{E^{\mathrm{op}}} = \theta_{E^{\mathrm{op}}}^{-1}$ is the generalized determinant, see Definition (4.5). We set

$$\begin{aligned}
\Xi_1 &= \mathrm{K}_1^{\mathrm{B}}(f_M)^{-1}(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))\ ,\\
\Xi_2 &= \mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}(\Xi_1)\ ,\\
\Xi_3 &= \eta_{E^{\mathrm{op}}}^{-1}(\Xi_2)\ ,\ \text{and}\\
\Xi_4 &= \det{}_{E^{\mathrm{op}}}(\Xi_3)\ .
\end{aligned}$$

With this notation, $\Xi_4 = \sigma^{-1}(\mathrm{Im}\,\mathrm{K}_1^{\mathrm{B}}(i))$. Thus, proving (7.7.3) amounts to showing that $\Xi_4 = \Xi$, which is done below.

As e_M is a quasi-inverse of f_M, see Proposition (5.2), $\mathrm{K}_1^{\mathrm{B}}(f_M)^{-1} = \mathrm{K}_1^{\mathrm{B}}(e_M)$. For a generator $[F_j, \varphi]$ of $\operatorname{Im} \mathrm{K}_1^{\mathrm{B}}(i)$ one has $\mathrm{K}_1^{\mathrm{B}}(e_M)([F_j, \varphi]) = [F_j M, \varphi_M]$, and thus:

– The subgroup Ξ_1 of $\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}}))$ is generated by elements $[F_j M, \varphi_M]$, where j ranges over $\{1, \dots, t\}$ and φ over all automorphisms of F_j.

Recall the choice made in Construction (2.8), that is, for every $j \in \{1, \dots, t\}$ and $\alpha \in \operatorname{Aut}_R(M_j)$ one has chosen $\beta_{j,\alpha} \in \operatorname{Aut}_R(X_j)$ and $\gamma_{j,\alpha} \in \operatorname{Aut}_R(\tau(M_j))$ such that the diagram (2.8.1) is commutative. Consider for each j and α the element

$$\begin{aligned}\xi_{j,\alpha} = {} & \big[\operatorname{Hom}_R(M, M_j), \operatorname{Hom}_R(M, \alpha)\big] - \big[\operatorname{Hom}_R(M, X_j), \operatorname{Hom}_R(M, \beta_{j,\alpha})\big] \\ & + \big[\operatorname{Hom}_R(M, \tau(M_j)), \operatorname{Hom}_R(M, \gamma_{j,\alpha})\big]\end{aligned}$$

in $\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$. We shall prove that:

– The subgroup Ξ_2 of $\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))$ is generated by the elements $\xi_{j,\alpha}$, where j ranges over $\{1, \dots, t\}$ and α over all automorphisms of M_j.

To this end, note that the proof of Bass' resolution theorem, see e.g. [16, thm. 3.1.14], gives a recipe for computing $\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}$ of an element $[Z, \psi] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}(E^{\mathrm{op}}))$. In the loop category $\Omega(\mathsf{mod}(E^{\mathrm{op}}))$, see (A.2), one takes any bounded and augmented projective resolution of (Z, ψ), that is, a commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & P_n \arrow[r, "\delta_n"] \arrow[d, "\psi_n", "\cong"'] & \cdots \arrow[r, "\delta_2"] & P_1 \arrow[r, "\delta_1"] \arrow[d, "\psi_1", "\cong"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\psi_0", "\cong"'] & Z \arrow[r] \arrow[d, "\psi", "\cong"'] & 0 \\ 0 \arrow[r] & P_n \arrow[r, "\delta_n"] & \cdots \arrow[r, "\delta_2"] & P_1 \arrow[r, "\delta_1"] & P_0 \arrow[r, "\varepsilon"] & Z \arrow[r] & 0\,, \end{tikzcd}$$

where ψ_i is an automorphism of $P_i \in \mathsf{proj}(E^{\mathrm{op}})$. Now one has

$$\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}([Z, \psi]) = \textstyle\sum_{i=0}^{n} (-1)^i [P_i, \psi_i] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}(E^{\mathrm{op}}))\,.$$

Now, recall from Theorem (6.2) that there is an exact sequence in $\mathsf{mod}(\mathsf{MCM}\,R)$,

$$0 \longrightarrow \operatorname{Hom}_R(-, \tau(M_j)) \longrightarrow \operatorname{Hom}_R(-, X_j) \longrightarrow \operatorname{Hom}_R(-, M_j) \longrightarrow F_j \longrightarrow 0\,.$$

Thus, the commutative diagram (2.8.1) in $\mathsf{MCM}\,R$ induces a commutative diagram in $\mathsf{mod}(\mathsf{MCM}\,R)$ with exact row(s),

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(-, \tau(M_j)) \arrow[r] \arrow[d, "\operatorname{Hom}_R(-,\gamma_{j,\alpha})", "\cong"'] & \operatorname{Hom}_R(-, X_j) \arrow[r] \arrow[d, "\operatorname{Hom}_R(-,\beta_{j,\alpha})", "\cong"'] & \operatorname{Hom}_R(-, M_j) \arrow[r] \arrow[d, "\operatorname{Hom}_R(-,\alpha)", "\cong"'] & F_j \arrow[r] \arrow[d, dashed, "\varphi", "\cong"'] & 0 \\ 0 \arrow[r] & \operatorname{Hom}_R(-, \tau(M_j)) \arrow[r] & \operatorname{Hom}_R(-, X_j) \arrow[r] & \operatorname{Hom}_R(-, M_j) \arrow[r] & F_j \arrow[r] & 0\,, \end{tikzcd}$$

where φ is the uniquely determined natural endotransformation of F_j that makes this diagram commutative. Note that φ is an automorphism by the Five Lemma, and thus $[F_j M, \varphi_M]$ belongs to Ξ_1. Evaluating the diagram above at $M = {}_{R,E}M$, we get a projective resolution of $[F_j M, \varphi_M]$ in $\Omega(\mathsf{mod}(E^{\mathrm{op}}))$, and hence

$$\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}([F_j M, \varphi_M]) = \xi_{j,\alpha}\,. \tag{7.7.4}$$

This proves that every $\xi_{j,\alpha}$ belongs to Ξ_2.

To prove that the elements $\xi_{j,\alpha}$ generate $\Xi_2 = \mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}(\Xi_1)$, it suffices to argue that for every generator $[F_j M, \varphi_M]$ of Ξ_1, where j is in $\{1, \dots, t\}$ and φ is an automorphism of F_j, there exists some α in $\operatorname{Aut}_R(M_j)$ such that (7.7.4) holds. Since the category $\mathsf{MCM}\,R$ is a Krull–Schmidt variety in the sense of Auslander [1, II, §2], it follows by [1, II, prop. 2.1(b,c)] and [1, I, prop. 4.7] that $\operatorname{Hom}_R(-, M_j) \to F_j \to 0$

is a projective cover in $\mathsf{mod}(\mathsf{MCM}\,R)$ in the sense of Definition (7.4). In particular, φ lifts to a natural transformation ψ of $\mathrm{Hom}_R(-,M_j)$, which must be an automorphism by Lemma (7.5). Thus we have a commutative diagram in $\mathsf{mod}(\mathsf{MCM}\,R)$,

$$\begin{tikzcd} \mathrm{Hom}_R(-,M_j) \arrow[r, two heads] \arrow[d, dashed, "\psi"', "\cong"] & F_j \arrow[d, "\cong"', "\varphi"] \\ \mathrm{Hom}_R(-,M_j) \arrow[r, two heads] & F_j\ . \end{tikzcd}$$

Since the Yoneda functor $y_M\colon \mathsf{MCM}\,R \to \mathsf{mod}(\mathsf{MCM}\,R)$ is fully faithful, see [21, lem. (4.3)], there exists a unique automorphism α of M_j such that $\psi = \mathrm{Hom}_R(-,\alpha)$. For this particular α, the arguments given above show that (7.7.4) holds.

Having described the generators $\xi_{j,\alpha}$ of Ξ_2, we turn to the group Ξ_3. Clearly, Ξ_3 is generated by the elements $\eta^{-1}_{E^{\mathrm{op}}}(\xi_{j,\alpha})$. By the definition of $\xi_{j,\alpha}$, by Lemma (7.6), and from the fact that $\eta^{-1}_{E^{\mathrm{op}}}$ is a group homomorphism, it follows that

$$\eta^{-1}_{E^{\mathrm{op}}}(\xi_{j,\alpha}) = \tilde{\alpha}^T(\tilde{\beta}^T_{j,\alpha})^{-1}\tilde{\gamma}^T_{j,\alpha} \in \mathrm{K}_1^{\mathrm{C}}(E^{\mathrm{op}})\ .$$

As Ξ_3 is generated by these elements, it is immediate from the definition of Ξ_4, and from the fact that $\det_{E^{\mathrm{op}}}$ is a homomorphism, that Ξ_4 is generated by the elements

$$(\det_{E^{\mathrm{op}}}\tilde{\alpha}^T)(\det_{E^{\mathrm{op}}}\tilde{\beta}^T_{j,\alpha})^{-1}(\det_{E^{\mathrm{op}}}\tilde{\gamma}^T_{j,\alpha}) = (\det_E\tilde{\alpha})(\det_E\tilde{\beta}_{j,\alpha})^{-1}(\det_E\tilde{\gamma}_{j,\alpha})\ ,$$

where the equality is by Lemma (4.7). Since these are exactly the generators of the group Ξ, see Definition (2.9), it follows that one has $\Xi_4 = \Xi$, as desired. □

8. Abelianization of Automorphism Groups

To apply Theorem (2.11), one must compute $\mathrm{Aut}_R(M)_{\mathrm{ab}}$, i.e. the abelianization of the automorphism group of the representation generator M. In Proposition (8.6) we compute $\mathrm{Aut}_R(M)_{\mathrm{ab}}$ for the R-module $M = R\oplus\mathfrak{m}$, which is a representation generator for $\mathsf{MCM}\,R$ if $\mathfrak{m}$ happens to be the only non-free indecomposable maximal Cohen–Macaulay module over R. Rings for which this is the case will be studied in Section 9. Throughout this section, A denotes any ring.

(8.1) **Definition.** Let $N_1,\ldots,N_s$ be A-modules, and set $N = N_1\oplus\cdots\oplus N_s$. We view elements in N as column vectors.

For $\varphi\in\mathrm{Aut}_A(N_i)$ we denote by $d_i(\varphi)$ the automorphism of N which has as its diagonal $1_{N_1},\ldots,1_{N_{i-1}},\varphi,1_{N_{i+1}},\ldots,1_{N_s}$ and 0 in all other entries.

For $i\neq j$ and $\mu\in\mathrm{Hom}_A(N_j,N_i)$ we denote by $e_{ij}(\mu)$ the automorphism of N with diagonal $1_{N_1},\ldots,1_{N_s}$, and whose only non-trivial off-diagonal entry is μ in position (i,j).

(8.2) **Lemma.** *Let $N_1,\ldots,N_s$ be A-modules and set $N = N_1\oplus\cdots\oplus N_s$. If $2\in A$ is a unit, if $i\neq j$, and if $\mu\in\mathrm{Hom}_A(N_j,N_i)$ then $e_{ij}(\mu)$ is a commutator in $\mathrm{Aut}_A(N)$.*

Proof. The commutator of φ and ψ in $\mathrm{Aut}_A(N)$ is $[\varphi,\psi]=\varphi\psi\varphi^{-1}\psi^{-1}$. It is easily verified that $e_{ij}(\mu) = [e_{ij}(\frac{\mu}{2}), d_j(-1_{N_j})]$ if $i\neq j$. □

The idea in the proof above is certainly not new. It appears, for example, already in Litoff [14, proof of thm. 2] in the case $s=2$.

(8.3) **Lemma.** *Let X and Y be non-isomorphic A-modules with local endomorphism rings. Let $\varphi,\psi\in\mathrm{End}_A(X)$ and assume that ψ factors through Y. Then one has $\psi\notin\mathrm{Aut}_A(X)$. Futhermore, $\varphi\in\mathrm{Aut}_A(X)$ if and only if $\varphi+\psi\in\mathrm{Aut}_A(X)$.*

Proof. Write $\psi = \psi''\psi'$ with $\psi'\colon X \to Y$ and $\psi''\colon Y \to X$. If ψ is an automorphism, then ψ'' is a split epimorphism and hence an isomorphism as Y is indecomposable. This contradicts the assumption that X and Y are not isomorphic. The second assertion now follows as $\mathrm{Aut}_A(X)$ is the set of units in the local ring $\mathrm{End}_A(X)$. □

(8.4) **Proposition.** *Let $N_1, \dots, N_s$ be pairwise non-isomorphic A-modules with local endomorphism rings. An endomorphism*

$$\alpha = (\alpha_{ij}) \in \mathrm{End}_A(N_1 \oplus \cdots \oplus N_s) \quad \textit{with} \quad \alpha_{ij} \in \mathrm{Hom}_A(N_j, N_i)$$

is an automorphism if and only if $\alpha_{11}, \alpha_{22}, \dots, \alpha_{ss}$ are automorphisms.

Furthermore, every α in $\mathrm{Aut}_A(N)$ can be written as a product of automorphisms of the form $d_i(\cdot)$ and $e_{ij}(\cdot)$, cf. Definition (8.1).

Proof. "Only if": Assume that $\alpha = (\alpha_{ij})$ is an automorphism with inverse $\beta = (\beta_{ij})$ and let $i = 1, \dots, s$ be given. In the local ring $\mathrm{End}_A(N_i)$ one has $1_{N_i} = \sum_{j=1}^{s} \alpha_{ij}\beta_{ji}$, and hence one of the terms $\alpha_{ij}\beta_{ji}$ must be an automorphism. As $\alpha_{ij}\beta_{ji}$ is not an automorphism for $j \neq i$, see Lemma (8.3), it follows that $\alpha_{ii}\beta_{ii}$ is an automorphism. Similarly, $\beta_{ii}\alpha_{ii}$ is an automorphism, and thus α_{ii} and β_{ii} are both automorphisms.

"If": By induction on $s \geqslant 1$. The assertion is trivial for $s = 1$. Now let $s > 1$. Assume that $\alpha_{11}, \alpha_{22}, \dots, \alpha_{ss}$ are automorphisms. Recall the notation from (8.1). By composing α with $e_{s1}(-\alpha_{s1}\alpha_{11}^{-1}) \cdots e_{31}(-\alpha_{31}\alpha_{11}^{-1})e_{21}(-\alpha_{21}\alpha_{11}^{-1})$ from the left and with $e_{12}(-\alpha_{11}^{-1}\alpha_{12})e_{13}(-\alpha_{11}^{-1}\alpha_{13}) \cdots e_{1s}(-\alpha_{11}^{-1}\alpha_{1s})$ from the right, one gets an endomorphism of the form

$$\alpha' = \left(\begin{array}{c|c} \alpha_{11} & 0 \\ \hline 0 & \beta \end{array}\right) = d_1(\alpha_{11}) \left(\begin{array}{c|c} 1_{N_1} & 0 \\ \hline 0 & \beta \end{array}\right),$$

where $\beta \in \mathrm{End}_A(N_2 \oplus \cdots \oplus N_s)$ is an $(s-1) \times (s-1)$ matrix with diagonal entries given by $\alpha_{jj} - \alpha_{j1}\alpha_{11}^{-1}\alpha_{1j}$ for $j = 2, \dots, s$. By applying Lemma (8.3) to the situation $\varphi = \alpha_{jj} - \alpha_{j1}\alpha_{11}^{-1}\alpha_{1j}$ and $\psi = \alpha_{j1}\alpha_{11}^{-1}\alpha_{1j}$, it follows that the diagonal entries in β are all automorphisms. By the induction hypothesis, β is now an automorphism and can be written as a product of automorphisms of the form $d_i(\cdot)$ and $e_{ij}(\cdot)$. Consequently, the same is true for α', and hence also for α. □

(8.5) **Corollary.** *Assume that $2 \in A$ is a unit and let $N_1, \dots, N_s$ be pairwise non-isomorphic A-modules with local endomorphism rings. The homomorphism,*

$$\Delta\colon \mathrm{Aut}_A(N_1) \times \cdots \times \mathrm{Aut}_A(N_s) \longrightarrow \mathrm{Aut}_A(N_1 \oplus \cdots \oplus N_s)\ ,$$

given by $\Delta(\varphi_1, \dots, \varphi_s) = d_1(\varphi_1) \cdots d_s(\varphi_s)$, induces a surjective homomorphism,

$$\Delta_{\mathrm{ab}}\colon \mathrm{Aut}_A(N_1)_{\mathrm{ab}} \oplus \cdots \oplus \mathrm{Aut}_A(N_s)_{\mathrm{ab}} \longrightarrow \mathrm{Aut}_A(N_1 \oplus \cdots \oplus N_s)_{\mathrm{ab}}\ .$$

Proof. By Proposition (8.4) every element in $\mathrm{Aut}_A(N_1 \oplus \cdots \oplus N_s)$ is a product of automorphisms of the form $d_i(\cdot)$ and $e_{ij}(\cdot)$. As $2 \in A$ is a unit, Lemma (8.2) yields that every element of the form $e_{ij}(\cdot)$ is a commutator; thus in $\mathrm{Aut}_A(N_1 \oplus \cdots \oplus N_s)_{\mathrm{ab}}$ every element is a product of elements of the form $d_i(\cdot)$, so Δ_{ab} is surjective. □

(8.6) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be any commutative local ring such that $2 \in R$ is a unit. Assume that $\mathfrak{m}$ is not isomorphic to R and that the endomorphism ring $\mathrm{End}_R(\mathfrak{m})$ is commutative and local. There is an isomorphism of abelian groups,*

$$\delta\colon \mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}} \xrightarrow{\cong} k^* \oplus \mathrm{Aut}_R(\mathfrak{m})\ ,$$

given by

$$\begin{pmatrix} \alpha_{11} & \alpha_{22} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \longmapsto \big([\alpha_{11}(1)]_{\mathfrak{m}}, \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}\big) \ .$$

Proof. First note that the image of any homomorphism $\alpha\colon \mathfrak{m} \to R$ is contained in $\mathfrak{m}$. Indeed if $\operatorname{Im}\alpha \nsubseteq \mathfrak{m}$, then $u = \alpha(a)$ is a unit for some $a \in \mathfrak{m}$, and thus $\alpha(u^{-1}a) = 1$. It follows that α is surjective, and hence a split epimorphism as R is free. Since $\mathfrak{m}$ is indecomposable, α must be an isomorphism, which is a contradiction.

Therefore, given an endomorphism,

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \operatorname{End}_R(R \oplus \mathfrak{m}) = \begin{pmatrix} \operatorname{Hom}_R(R,R) & \operatorname{Hom}_R(\mathfrak{m},R) \\ \operatorname{Hom}_R(R,\mathfrak{m}) & \operatorname{Hom}_R(\mathfrak{m},\mathfrak{m}) \end{pmatrix},$$

we may by (co)restriction view the entries α_{ij} as elements in the endomorphism ring $\operatorname{End}_R(\mathfrak{m})$. As this ring is assumed to be commutative, the determinant map

$$\operatorname{End}_R(R \oplus \mathfrak{m}) \longrightarrow \operatorname{End}_R(\mathfrak{m}) \qquad \text{given by} \qquad (\alpha_{ij}) \longmapsto \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$$

preserves multiplication. If $(\alpha_{ij}) \in \operatorname{Aut}_R(R \oplus \mathfrak{m})$, then Proposition (8.4) implies that $\alpha_{11} \in \operatorname{Aut}_R(R)$ and $\alpha_{22} \in \operatorname{Aut}_R(\mathfrak{m})$, and thus $\alpha_{11}\alpha_{22} \in \operatorname{Aut}_R(\mathfrak{m})$. By applying Lemma (8.3) to $\varphi = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$ and $\psi = \alpha_{21}\alpha_{12}$ we get $\varphi \in \operatorname{Aut}_R(\mathfrak{m})$, and hence the determinant map is a group homomorphism $\operatorname{Aut}_R(R \oplus \mathfrak{m}) \to \operatorname{Aut}_R(\mathfrak{m})$.

The map $\operatorname{Aut}_R(R \oplus \mathfrak{m}) \to k^*$ defined by $(\alpha_{ij}) \mapsto [\alpha_{11}(1)]_{\mathfrak{m}}$ is also a group homomorphism. Indeed, entry $(1,1)$ in the product $(\alpha_{ij})(\beta_{ij})$ is $\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21}$. Here α_{12} is a homomorphism $\mathfrak{m} \to R$, and hence $\alpha_{12}\beta_{21}(1) \in \mathfrak{m}$ by the arguments in the beginning of the proof. Consequently one has

$$[(\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21})(1)]_{\mathfrak{m}} = [(\alpha_{11}\beta_{11})(1)]_{\mathfrak{m}} = [\alpha_{11}(1)\beta_{11}(1)]_{\mathfrak{m}} = [\alpha_{11}(1)]_{\mathfrak{m}}[\beta_{11}(1)]_{\mathfrak{m}} \ .$$

These arguments and the fact that the groups k^* and $\operatorname{Aut}_R(\mathfrak{m})$ are abelian show that the map δ described in the proposition is a well-defined group homomorphism. Evidently, δ is surjective; indeed, for $[r]_{\mathfrak{m}} \in k^*$ and $\varphi \in \operatorname{Aut}_R(\mathfrak{m})$ one has

$$\delta \begin{pmatrix} r1_R & 0 \\ 0 & r^{-1}\varphi \end{pmatrix} = ([r]_{\mathfrak{m}}, \varphi) \ .$$

To show that δ is injective, assume that $\alpha \in \operatorname{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}$ with $\delta(\alpha) = ([1]_{\mathfrak{m}}, 1_{\mathfrak{m}})$. By Corollary (8.5) we can assume that $\alpha = (\alpha_{ij})$ is a diagonal matrix. We write $\alpha_{11} = r1_R$ for some unit $r \in R$. Since one has $\delta(\alpha) = ([r]_{\mathfrak{m}}, r\alpha_{22})$ we conclude that $r \in 1 + \mathfrak{m}$ and $\alpha_{22} = r^{-1}1_{\mathfrak{m}}$, that is, α has the form

$$\alpha = \begin{pmatrix} r1_R & 0 \\ 0 & r^{-1}1_{\mathfrak{m}} \end{pmatrix} \qquad \text{with} \qquad r \in 1 + \mathfrak{m} \ .$$

Thus, proving injectivity of δ amounts to showing that every automorphism α of the form above belongs to the commutator subgroup of $\operatorname{Aut}_R(R \oplus \mathfrak{m})$. As $r - 1 \in \mathfrak{m}$ the map $(r-1)1_R$ gives a homomorphism $R \to \mathfrak{m}$. Since $r(r^{-1} - 1) = 1 - r \in \mathfrak{m}$ and $r \notin \mathfrak{m}$, it follows that $r^{-1} - 1 \in \mathfrak{m}$. Thus $(r^{-1} - 1)1_R$ gives another homomorphism $R \to \mathfrak{m}$. If $\iota\colon \mathfrak{m} \hookrightarrow R$ denotes the inclusion, then one has[1]

$$\begin{pmatrix} r1_R & 0 \\ 0 & r^{-1}1_{\mathfrak{m}} \end{pmatrix} = \begin{pmatrix} 1_R & 0 \\ (r^{-1}-1)1_R & 1_{\mathfrak{m}} \end{pmatrix} \begin{pmatrix} 1_R & \iota \\ 0 & 1_{\mathfrak{m}} \end{pmatrix} \begin{pmatrix} 1_R & 0 \\ (r-1)1_R & 1_{\mathfrak{m}} \end{pmatrix} \begin{pmatrix} 1_R & -r^{-1}\iota \\ 0 & 1_{\mathfrak{m}} \end{pmatrix}.$$

The right-hand of this equality is a product of matrices of the form $e_{ij}(\cdot)$, and since $2 \in R$ is a unit the desired conclusion now follows from Lemma (8.2). $\square$

[1] The identity comes from the standard proof of Whitehead's lemma; see e.g. [18, lem. (1.4)].

9. Examples

(9.1) **Example.** If R is regular, then there are isomorphisms,

$$\mathrm{K}_1(\mathsf{mod}\, R) \cong \mathrm{K}_1(\mathsf{proj}\, R) \cong \mathrm{K}_1^{\mathrm{C}}(R) \cong R^* \,.$$

The first isomorphism is by Quillen's resolution theorem [15, §4 thm. 3], the second one is mentioned in (A.6), and the third one is well-known; see e.g. [18, exa. (1.6)]. Theorem (2.11) confirms this result, indeed, as $M = R$ is a representation generator for $\mathsf{MCM}\, R = \mathsf{proj}\, R$ one has $\mathrm{Aut}_R(M)_{\mathrm{ab}} = R^*$. As there are no Auslander–Reiten sequences in this case, the subgroup Ξ is generated by the empty set, so $\Xi = 0$.

We now illustrate how Theorem (2.11) applies to compute $\mathrm{K}_1(\mathsf{mod}\, R)$ for the ring $R = k[X]/(X^2)$. The answer is well-known to be k^*, indeed, for any commutative artinian local ring R with residue field k one has $\mathrm{K}_1(\mathsf{mod}\, R) \cong k^*$ by [15, §5 cor. 1].

(9.2) **Example.** *Let $R = k[X]/(X^2)$ be the ring of dual numbers over a field k with* $\mathrm{char}(k) \neq 2$. *Denote by* $\mathrm{inc}\colon \mathsf{proj}\, R \to \mathsf{mod}\, R$ *the inclusion functor. The homomorphism* $\mathrm{K}_1(\mathrm{inc})$ *may be identified with the map,*

$$\mu\colon R^* \longrightarrow k^* \qquad \text{given by} \qquad a + bX \longmapsto a^2 \,.$$

Proof. It is well-known that the maximal ideal $\mathfrak{m} = (X)$ is the only non-free indecomposable Cohen–Macaulay R-module, so $R \oplus \mathfrak{m}$ is a representation generator for $\mathsf{MCM}\, R$. There is an isomorphism $k \to \mathrm{End}_R(\mathfrak{m})$ of R-algebras given by $a \mapsto a1_{\mathfrak{m}}$, in particular, $\mathrm{End}_R(\mathfrak{m})$ is commutative. Via this isomorphism, k^* corresponds to $\mathrm{Aut}_R(\mathfrak{m})$. The Auslander–Reiten sequence ending in $\mathfrak{m}$ is

$$0 \longrightarrow \mathfrak{m} \xrightarrow{\iota} R \xrightarrow{X} \mathfrak{m} \longrightarrow 0 \,,$$

where ι is the inclusion. The Auslander–Reiten homomorphism $\Upsilon = \binom{-1}{2}\colon \mathbb{Z} \to \mathbb{Z}^2$ is injective, so Theorem (2.11) can be applied. Note that for every $a1_{\mathfrak{m}} \in \mathrm{Aut}_R(\mathfrak{m})$, where $a \in k^*$, there is a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m} \arrow[r, "\iota"] \arrow[d, "a1_{\mathfrak{m}}", "\cong"'] & R \arrow[r, "X"] \arrow[d, "a1_R", "\cong"'] & \mathfrak{m} \arrow[r] \arrow[d, "a1_{\mathfrak{m}}", "\cong"'] & 0 \\
0 \arrow[r] & \mathfrak{m} \arrow[r, "\iota"] & R \arrow[r, "X"] & \mathfrak{m} \arrow[r] & 0 \,.
\end{tikzcd}$$

Applying the tilde construction (2.5) to the automorphisms $a1_{\mathfrak{m}}$ and $a1_R$ one gets

$$\widetilde{a1_{\mathfrak{m}}} = \begin{pmatrix} 1_R & 0 \\ 0 & a1_{\mathfrak{m}} \end{pmatrix} \qquad \text{and} \qquad \widetilde{a1_R} = \begin{pmatrix} a1_R & 0 \\ 0 & 1_{\mathfrak{m}} \end{pmatrix};$$

see Example (2.6). In view of Definition (2.9) and Remark (2.10), the subgroup Ξ of $\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}$ is therefore generated by all elements of the form

$$\xi_a := (\widetilde{a1_{\mathfrak{m}}})(\widetilde{a1_R})^{-1}(\widetilde{a1_{\mathfrak{m}}}) = \begin{pmatrix} a^{-1}1_R & 0 \\ 0 & a^2 1_{\mathfrak{m}} \end{pmatrix} \qquad \text{where} \qquad a \in k^* \,.$$

Denote by ω the composite of the isomorphisms,

$$\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}} \xrightarrow[\cong]{\delta} k^* \oplus \mathrm{Aut}_R(\mathfrak{m}) \xrightarrow[\cong]{} k^* \oplus k^* \,,$$

where δ is the isomorphism from Proposition (8.6). As $\omega(\xi_a) = (a^{-1}, a)$ we get that $\omega(\Xi) = \{(a^{-1}, a) \,|\, a \in k^*\}$ and thus ω induces the first group isomorphism below,

$$\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\Xi \xrightarrow[\cong]{\varpi} (k^* \oplus k^*)/\omega(\Xi) \xrightarrow[\cong]{\chi} k^* \,;$$

the second isomorphism is induced by the surjective homomorphism $k^* \oplus k^* \to k^*$, given by $(b,a) \mapsto ba$, whose kernel is exactly $\omega(\Xi)$. In view of Theorem (2.11) and the isomorphisms ϖ and χ above, it follows that $\mathrm{K}_1(\mathsf{mod}\,R) \cong k^*$.

Theorem (2.11) asserts that $\mathrm{K}_1(\mathrm{inc})$ may be identified with the homomorphism

$$\lambda\colon R^* \longrightarrow \mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\Xi \qquad \text{given by} \qquad r \longmapsto \begin{pmatrix} r1_R & 0 \\ 0 & 1_{\mathfrak{m}} \end{pmatrix}.$$

It remains to note that the isomorphism $\chi\varpi$ identifies λ with the homomorphism μ described in the example, indeed, one has $\chi\varpi\lambda = \mu$. □

Example (9.2) shows that for $R = k[X]/(X^2)$ the canonical homomorphism,

$$R^* \cong \mathrm{K}_1(\mathsf{proj}\,R) \xrightarrow{\mathrm{K}_1(\mathrm{inc})} \mathrm{K}_1(\mathsf{mod}\,R) \cong k^*$$

is not an isomorphism. It turns out that if k is algebraically closed with characteristic zero, then there exists a non-canonical isomorphism between R^* and k^*.

(9.3) **Proposition.** *Let $R = k[X]/(X^2)$ where k is an algebraically closed field with characteristic $p \geqslant 0$. The following assertions hold.*

(a) *If $p > 0$, then the groups R^* and k^* are not isomorphic.*
(b) *If $p = 0$, then there exists a (non-canonical) group isomorphism $R^* \cong k^*$.*

Proof. There is a group isomorphism $R^* \to k^* \oplus k^+$ given by $a + bX \mapsto (a, b/a)$, where k^+ denotes the underlying abelian group of the field k.

"(a)": Let $\varphi = (\varphi_1, \varphi_2)\colon k^* \to k^* \oplus k^+$ be any group homomorphism. As k is algebraically closed, every element in $x \in k^*$ has the form $x = y^p$ for some $y \in k^*$. Therefore $\varphi(x) = \varphi(y^p) = \varphi(y)^p = (\varphi_1(y), \varphi_2(y))^p = (\varphi_1(y)^p, p\varphi_2(y)) = (\varphi_1(x), 0)$, which shows that φ is not surjective.

"(b)": Since $p = 0$ the abelian group k^+ is divisible and torsion free. Therefore $k^+ \cong \mathbb{Q}^{(I)}$ for some index set I. There exist algebraic field extensions of $\mathbb{Q}$ of any finite degree, and these are all contained in the algebraically closed field k. Thus $|I| = \dim_{\mathbb{Q}} k$ must be infinite, and it follows that $|I| = |k|$.

The abelian group k^* is also divisible, but it has torsion. Write $k^* \cong T \oplus (k^*/T)$, where $T = \{x \in k^* \,|\, \exists\, n \in \mathbb{N}\colon x^n = 1\}$ is the torsion subgroup of k^*. For the divisible torsion free abelian group k^*/T one has $k^*/T \cong \mathbb{Q}^{(J)}$ for some index set J. It is not hard to see that $|J|$ must be infinite, and hence $|J| = |k^*/T|$. As $|T| = \aleph_0$ it follows that $|k| = |k^*| = \aleph_0 + |J| = |J|$.

Since $|J| = |k| = |I|$ one gets $k^* \cong T \oplus \mathbb{Q}^{(J)} \cong T \oplus \mathbb{Q}^{(J)} \oplus \mathbb{Q}^{(I)} \cong k^* \oplus k^+$. □

The artinian ring $R = k[X]/(X^2)$ from Example (9.2) has length $\ell = 2$ and this power is also involved in the description of the homomorphism $\mu = \mathrm{K}_1(\mathrm{inc})$. The next result, which might be well-known to experts, shows that this is no coincidence.

(9.4) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be a commutative artinian local ring of length ℓ. The group homomorphism $R^* \cong \mathrm{K}_1(\mathsf{proj}\,R) \to \mathrm{K}_1(\mathsf{mod}\,R) \cong k^*$ induced by the inclusion* $\mathrm{inc}\colon \mathsf{proj}\,R \to \mathsf{mod}\,R$ *is the composition of the homomorphisms,*

$$R^* \xrightarrow{\pi} k^* \xrightarrow{(\cdot)^\ell} k^* \,,$$

where $\pi\colon R \twoheadrightarrow R/\mathfrak{m} = k$ is the canonical quotient map and $(\cdot)^\ell$ is the ℓ'th power.

Proof. By Theorem (A.7) the Gersten–Sherman transformation $\zeta_{\mathsf{proj}\,R}$ is an isomorphism. Since $\mathsf{mod}\,R$ is a length category with only one simple object, namely k, Theorem (A.8) shows that $\zeta_{\mathsf{mod}\,R}$ is an isomorphism as well. Hence, to prove the proposition we may work with Bass' K_1-group instead of Quillen's.

Denote by $(\mathsf{mod}\,R)_{\mathrm{ss}}$ the abelian category of semisimple objects in $\mathsf{mod}\,R$, which may be identified with $\mathsf{proj}\,k$. By Bass' devissage theorem [6, VIII§3 thm. (3.4)], the inclusion $\jmath\colon (\mathsf{mod}\,R)_{\mathrm{ss}} \hookrightarrow \mathsf{mod}\,R$ induces an isomorphism $\mathrm{K}_1^{\mathrm{B}}(\jmath)$. Denote by μ the composition of the isomorphisms,

$$\mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R) \xrightarrow[\cong]{\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}} \mathrm{K}_1^{\mathrm{B}}((\mathsf{mod}\,R)_{\mathrm{ss}}) = \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,k)) \xrightarrow[\cong]{\eta_k^{-1}} \mathrm{K}_1^{\mathrm{C}}(k) \xrightarrow[\cong]{\det} k^* \ ;$$

here η_k is defined in (A.5), and det is the usual determinant map.

To compute $\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}$ of an element $[M,\alpha] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R)$ one takes any filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_i/M_{i-1} \in (\mathsf{mod}\,R)_{\mathrm{ss}}$ and $\alpha(M_i) \subseteq M_i$ for all i (e.g. a socle filtration). Now α induces an automorphism α_i of each composition factor M_i/M_{i-1} and one has

$$\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}([M,\alpha]) = \textstyle\sum_{i=1}^{n}[M_i/M_{i-1},\alpha_i] \in \mathrm{K}_1^{\mathrm{B}}((\mathsf{mod}\,R)_{\mathrm{ss}}) \ .$$

To prove the proposition, we argue that the following diagram is commutative,

$$\begin{tikzcd}
R^* \arrow[r, "\pi"] \arrow[d, "\rho_R"', "\cong"] & k^* \arrow[r, "(\cdot)^\ell"] & k^* \\
\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,R) \arrow[rr, "\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})"] & & \mathrm{K}_1^{\mathrm{B}}(\mathsf{mod}\,R) \arrow[u, "\cong"', "\mu"]
\end{tikzcd} ;$$

here ρ_R is the isomorphism from (7.7). For $r \in R^*$ one has $\mathrm{K}_1^{\mathrm{B}}(\mathrm{inc})\rho_R(r) = [R, r1_R]$. Take a composition series $0 = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_\ell = R$. The automorphism $\alpha = r1_R$ of R induces the automorphism $\alpha_i = \pi(r)1_k$ on each composition factor $\mathfrak{a}_i/\mathfrak{a}_{i-1} \cong k$, and therefore $\mathrm{K}_1^{\mathrm{B}}(\jmath)^{-1}([R,r1_R]) = \ell \cdot \big[k,\pi(r)1_k\big] = \big[k,\pi(r)^\ell 1_k\big]$. It follows that one has $\mu([R,r1_R]) = \pi(r)^\ell$, as desired. □

Our next example is a non-artinian ring, namely the simple curve singularity of type (A_2) studied by Yoshino in [21, prop. (5.11)].

(9.5) **Example.** *Let* $R = k[\![T^2,T^3]\!]$ *where* k *is an algebraically closed field with* $\mathrm{char}(k) \neq 2$. *Denote by* $\mathrm{inc}\colon \mathsf{proj}\,R \to \mathsf{mod}\,R$ *the inclusion functor. The homomorphism* $\mathrm{K}_1(\mathrm{inc})$ *may be identified with the inclusion map,*

$$\mu\colon R^* = k[\![T^2,T^3]\!]^* \hookrightarrow k[\![T]\!]^* \ .$$

Proof. By Yoshino [21, prop. (5.11)] the maximal ideal $\mathfrak{m} = (T^2,T^3)$ is the only non-free indecomposable Cohen–Macaulay R-module, so $R \oplus \mathfrak{m}$ is a representation generator for $\mathsf{MCM}\,R$. It is not hard to see that with

$$\partial = \begin{pmatrix} T^3 & T^4 \\ -T^2 & -T^3 \end{pmatrix}$$

one has the following augmented minimal free resolution of $\mathfrak{m}$,

$$\cdots \xrightarrow{\ \partial\ } R^2 \xrightarrow{\ \partial\ } R^2 \xrightarrow{\ \partial\ } R^2 \xrightarrow{(T^2\ T^3)} \mathfrak{m} \longrightarrow 0 \ . \qquad (9.5.1)$$

Even though T is not an element in the ring $R = k[\![T^2, T^3]\!]$, multiplication by T is a well-defined endomorphism of $\mathfrak{m}$. Thus there is a ring homomorphism,

$$\chi\colon k[\![T]\!] \longrightarrow \mathrm{End}_R(\mathfrak{m}) \qquad \text{given by} \qquad h \longmapsto h1_{\mathfrak{m}}\ .$$

We claim that χ is an isomorphism. To show that χ is surjective, let $\alpha \in \mathrm{End}_R(\mathfrak{m})$ be given. As one has $T^2\alpha(T^3) - T^3\alpha(T^2) = 0$, i.e. $(\alpha(T^3)\ \ -\alpha(T^2))^t \in \mathrm{Ker}(T^2\ \ T^3)$, exactness of (9.5.1) yields elements $f, g \in R$ with

$$\begin{pmatrix} \alpha(T^3) \\ -\alpha(T^2) \end{pmatrix} = \partial \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} fT^3 + gT^4 \\ -fT^2 - gT^3 \end{pmatrix} = \begin{pmatrix} (f+gT)T^3 \\ -(f+gT)T^2 \end{pmatrix}, \tag{9.5.2}$$

and hence $\alpha = (f+gT)1_{\mathfrak{m}}$. To show injectivity of χ, assume that $h \in k[\![T]\!]$ satisfies $h1_{\mathfrak{m}} = 0$. Write $h = f + gT$ for some $f, g \in R$; for example, if $h = \sum_{n\geqslant 0} h_n T^n$ then $f = h - h_1 T$ and $g = h_1$ is an option. The computation in (9.5.2) shows that $(f\ \ g)^t$ belongs to $\mathrm{Ker}\,\partial$. By exactness of (9.5.1) this means that $(f\ \ g)^t \in \mathrm{Im}\,\partial$, and hence there exist $u, v \in R$ such that

$$\begin{pmatrix} f \\ g \end{pmatrix} = \partial \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uT^3 + vT^4 \\ -uT^2 - vT^3 \end{pmatrix}.$$

Consequently, $h = f + gT = (uT^3 + vT^4) - (uT^2 + vT^3)T = 0$.

We note that χ induces a group isomorphism $\chi\colon k[\![T]\!]^* \to \mathrm{Aut}_R(\mathfrak{m})$.

The Auslander–Reiten sequence ending in $\mathfrak{m}$ is

$$0 \longrightarrow \mathfrak{m} \xrightarrow{(1\ \ -T)^t} R \oplus \mathfrak{m} \xrightarrow{(T^2\ \ T)} \mathfrak{m} \longrightarrow 0\ .$$

We regard elements in $R \oplus \mathfrak{m}$ as column vectors. Let $\alpha = h1_{\mathfrak{m}} \in \mathrm{Aut}_R(\mathfrak{m})$, where $h \in k[\![T]\!]^*$, be given. Write $h = f + gT$ for some $f \in R^*$ and $g \in R$. It is straightforward to verify that there is a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m} \arrow[r, "(1\ \ -T)^t"] \arrow[d, "\gamma = (f-gT)1_{\mathfrak{m}}", "\cong"'] & R \oplus \mathfrak{m} \arrow[r, "(T^2\ \ T)"] \arrow[d, "\beta = \begin{pmatrix} f & g \\ gT^2 & f \end{pmatrix}", "\cong"'] & \mathfrak{m} \arrow[r] \arrow[d, "\alpha = (f+gT)1_{\mathfrak{m}}", "\cong"'] & 0 \\
0 \arrow[r] & \mathfrak{m} \arrow[r, "(1\ \ -T)^t"'] & R \oplus \mathfrak{m} \arrow[r, "(T^2\ \ T)"'] & \mathfrak{m} \arrow[r] & 0\ .
\end{tikzcd}$$

Note that β really is an automorphism; indeed, its inverse is given by

$$\beta^{-1} = (f^2 - g^2T^2)^{-1} \begin{pmatrix} f & -g \\ -gT^2 & f \end{pmatrix}.$$

We now apply the tilde construction (2.5) to α, β, and γ; by Example (2.6) we get:

$$\tilde{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & f+gT \end{pmatrix}, \quad \tilde{\beta} = \beta, \quad \text{and} \quad \tilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & f-gT \end{pmatrix}.$$

In view of Definition (2.9) and Remark (2.10), the subgroup Ξ of $\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}$ is therefore generated by all the elements

$$\xi_h := \tilde{\alpha}\tilde{\beta}^{-1}\tilde{\gamma} = (f^2 - g^2T^2)^{-1} \begin{pmatrix} f & -g(f-gT) \\ -gT^2(f+gT) & f(f^2 - g^2T^2) \end{pmatrix}.$$

Denote by ω the composite of the isomorphisms,

$$\mathrm{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}} \xrightarrow[\cong]{\delta} k^* \oplus \mathrm{Aut}_R(\mathfrak{m}) \xrightarrow[\cong]{1 \oplus \chi^{-1}} k^* \oplus k[\![T]\!]^*\ ,$$

where δ is the isomorphism from Proposition (8.6). Note that $\delta(\xi_h) = ([f]_{\mathfrak{m}}, 1_{\mathfrak{m}}) = (h(0), 1_{\mathfrak{m}})$ and hence $\omega(\xi_h) = (h(0), 1)$. It follows that $\omega(\Xi) = k^* \oplus \{1\}$ and thus ω induces a group isomorphism,

$$\varpi\colon \operatorname{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\Xi \xrightarrow{\cong} (k^* \oplus k[\![T]\!]^*)/\omega(\Xi) = k[\![T]\!]^* .$$

In view of this isomorphism, Theorem (2.11) shows that $\mathrm{K}_1(\mathsf{mod}\, R) \cong k[\![T]\!]^*$. Theorem (2.11) also asserts that $\mathrm{K}_1(\mathrm{inc})$ may be identified with the homomorphism

$$\lambda\colon R^* \longrightarrow \operatorname{Aut}_R(R \oplus \mathfrak{m})_{\mathrm{ab}}/\Xi \qquad \text{given by} \qquad f \longmapsto \begin{pmatrix} f1_R & 0 \\ 0 & 1_{\mathfrak{m}} \end{pmatrix}.$$

It remains to note that the isomorphism ϖ identifies λ with the inclusion map μ described in the example, indeed, one has $\varpi\lambda = \mu$. □

(9.6) **Remark.** It is also possible to compute $\mathrm{K}_1(\mathsf{mod}\, R)$ for the ring $R = k[\![T^2, T^3]\!]$ without using Theorem (2.11). Indeed, one has $R[T] = R[\![T]\!] = k[\![T]\!]$ and hence

$$\mathrm{K}_1(\mathsf{mod}\, R) \cong \mathrm{K}_1(\mathsf{mod}\, R[T]) \cong \mathrm{K}_1(\mathsf{mod}\, k[\![T]\!]) \cong k[\![T]\!]^* .$$

Here the first isomorphism is by [15, §6 thm. 8(i)] (a celebrated result in G-theory) and the last one follows as $k[\![T]\!]$ is regular; cf. Example (9.1).

Appendix A. The Gersten–Sherman Transformation

In the following, the Grothendieck group functor is denoted by G.

(A.1) Let A be a unital ring.

The *classical* K_0*-group* of A is defined as $\mathrm{K}_0^{\mathrm{C}}(A) = \mathrm{G}(\mathsf{proj}\, A)$, that is, the Grothendieck group of the category of finitely generated projective A-modules.

The *classical* K_1*-group* of A is defined as $\mathrm{K}_1^{\mathrm{C}}(A) = \mathrm{GL}(A)_{\mathrm{ab}}$, that is, the abelianization of the infinite (or stable) general linear group; see e.g. Bass [6, chap. V].

(A.2) Let $\mathcal{C}$ be any category. Its *loop category* $\Omega\mathcal{C}$ is the category whose objects are pairs (C, α) with $C \in \mathcal{C}$ and $\alpha \in \operatorname{Aut}_{\mathcal{C}}(C)$. A morphism $(C, \alpha) \to (C', \alpha')$ in $\Omega\mathcal{C}$ is a commutative diagram in $\mathcal{C}$,

$$\begin{array}{ccc} C & \xrightarrow{\psi} & C' \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\alpha'} \\ C & \xrightarrow{\psi} & C'. \end{array}$$

(A.3) Let $\mathcal{C}$ be a skeletally small exact category. Its loop category $\Omega\mathcal{C}$ is also skeletally small, and it inherits a natural exact structure from $\mathcal{C}$. *Bass'* K_1*-group* (also called *Bass' universal determinant group*) of $\mathcal{C}$, which we denote by $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$, is the Grothendieck group of $\Omega\mathcal{C}$, that is $\mathrm{G}(\Omega\mathcal{C})$, modulo the subgroup generated by all elements of the form

$$(C, \alpha) + (C, \alpha) - (C, \alpha\beta) ,$$

where $C \in \mathcal{C}$ and $\alpha, \beta \in \operatorname{Aut}_{\mathcal{C}}(C)$; see the book of Bass [6, chap. VIII§1] or Rosenberg [16, def. 3.1.6]. For (C, α) in $\Omega\mathcal{C}$ we denote by $[C, \alpha]$ its image in $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$.

(A.4) For every C in $\mathcal{C}$ one has $[C, 1_C] + [C, 1_C] = [C, 1_C 1_C] = [C, 1_C]$ in $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$. Consequently, $[C, 1_C]$ is the neutral element in $\mathrm{K}_1^{\mathrm{B}}(\mathcal{C})$.

(A.5) For a unital ring A there is by [16, thm. 3.1.7] a natural isomorphism,

$$\eta_A \colon \mathrm{K}_1^{\mathrm{C}}(A) \xrightarrow{\cong} \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A)\;.$$

The isomorphism η_A maps $\xi \in \mathrm{GL}_n(A)$, to the class $[A^n, \xi] \in \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A)$. Here ξ is viewed as an automorphism of the row space A^n (a free left A-module), that is, ξ acts by multiplication from the right.

The inverse map η_A^{-1} acts as follows. Let $[P, \alpha]$ be in $\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A)$. Choose any Q in $\mathsf{proj}\,A$ and any isomorphism $\psi \colon P \oplus Q \to A^n$ with $n \in \mathbb{N}$. In $\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A)$ one has

$$[P, \alpha] = [P, \alpha] + [Q, 1_Q] = [P \oplus Q, \alpha \oplus 1_Q] = [A^n, \psi(\alpha \oplus 1_Q)\psi^{-1}]\;.$$

The automorphism $\psi(\alpha \oplus 1_Q)\psi^{-1}$ of (the row space) A^n can be identified with a matrix in $\beta \in \mathrm{GL}_n(A)$. The action of η_A^{-1} on $[P, \alpha]$ is now β's image in $\mathrm{K}_1^{\mathrm{C}}(A)$.

(A.6) Quillen defines in [15] functors $\mathrm{K}_n^{\mathrm{Q}}$ from the category of skeletally small exact categories to the category of abelian groups. More precisely, $\mathrm{K}_n^{\mathrm{Q}}(\mathcal{C}) = \pi_{n+1}(\mathrm{BQ}\mathcal{C}, 0)$ where Q is Quillen's Q-construction and B denotes the classifying space.

The functor $\mathrm{K}_0^{\mathrm{Q}}$ is naturally isomorphic to the Grothendieck group functor G; see [15, §2 thm. 1]. For a ring A there is a natural isomorphism $\mathrm{K}_1^{\mathrm{Q}}(\mathsf{proj}\,A) \cong \mathrm{K}_1^{\mathrm{C}}(A)$; see for example Srinivas [18, cor. (2.6) and thm. (5.1)].

Gersten sketches in [11, §5] a construction a natural transformation $\zeta \colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1^{\mathrm{Q}}$ of functors on the category of skeletally small exact categories. The details of this construction were later given by Sherman [17, §3], and for this reason we refer to ζ as the *Gersten–Sherman transformation*[2]. Examples due to Gersten and Murthy [11, prop. 5.1 and 5.2] show that for a general skeletally small exact category $\mathcal{C}$, the homomorphism $\zeta_{\mathcal{C}} \colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C})$ is neither injective nor surjective. For the exact category $\mathsf{proj}\,A$, where A is a ring, it is known that $\mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A)$ and $\mathrm{K}_1^{\mathrm{Q}}(\mathsf{proj}\,A)$ are isomorphic, indeed, thet are both isomorphic to the classical K-group $\mathrm{K}_1^{\mathrm{C}}(A)$; see (A.5) and (A.6). Therefore, a natural question arises: is $\zeta_{\mathsf{proj}\,A}$ an isomorphism? Sherman answers this question affirmatively in [17, pp. 231–232]; in fact, in *loc. cit.* Theorem 3.3 it is proved that $\zeta_{\mathcal{C}}$ is an isomorphism for every semisimple exact category, that is, an exact category in which every short exact sequence splits. We note these results of Gersten and Sherman for later use.

(A.7) **Theorem.** *There exists a natural transformation $\zeta \colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1^{\mathrm{Q}}$, which we call the Gersten–Sherman transformation, of functors on the category of skeletally small exact categories such that $\zeta_{\mathsf{proj}\,A} \colon \mathrm{K}_1^{\mathrm{B}}(\mathsf{proj}\,A) \to \mathrm{K}_1^{\mathrm{Q}}(\mathsf{proj}\,A)$ is an isomorphism for every ring A.* $\square$

We will also need the next result on the Gersten–Sherman transformation. Recall that a *length category* is an abelian category in which every object has finite length.

(A.8) **Theorem.** *If $\mathcal{A}$ is a skeletally small length category with only finitely many simple objects (up to isomorphism), then $\zeta_{\mathcal{A}} \colon \mathrm{K}_1^{\mathrm{B}}(\mathcal{A}) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{A})$ is an isomorphism.*

Proof. We begin with a general observation. Given skeletally small exact categories $\mathcal{C}_1$ and $\mathcal{C}_2$, there are exact projection functors $p_j \colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_j$ $(j = 1, 2)$. From the "elementary properties" of Quillen's K-groups listed in [15, §2], it follows that

[2] In the papers by Gersten [11] and Sherman [17], the functor $\mathrm{K}_1^{\mathrm{B}}$ is denoted by $\mathrm{K}_1^{\det}$.

the homomorphism $(\mathrm{K}_1^{\mathrm{Q}}(p_1), \mathrm{K}_1^{\mathrm{Q}}(p_2))\colon \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_1 \times \mathcal{C}_2) \to \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_1) \oplus \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_2)$ is an isomorphism. A similar argument shows that $(\mathrm{K}_1^{\mathrm{B}}(p_1), \mathrm{K}_1^{\mathrm{B}}(p_2))$ is an isomorphism. Since $\zeta\colon \mathrm{K}_1^{\mathrm{B}} \to \mathrm{K}_1^{\mathrm{Q}}$ is a natural transformation, there is a commutative diagram,

$$\begin{tikzcd} \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}_1 \times \mathcal{C}_2) \arrow[r, "(\mathrm{K}_1^{\mathrm{B}}(p_1),\mathrm{K}_1^{\mathrm{B}}(p_2))", "\cong"'] \arrow[d, "\zeta_{\mathcal{C}_1 \times \mathcal{C}_2}"'] & \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}_1) \oplus \mathrm{K}_1^{\mathrm{B}}(\mathcal{C}_2) \arrow[d, "\zeta_{\mathcal{C}_1} \oplus \zeta_{\mathcal{C}_2}"] \\ \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_1 \times \mathcal{C}_2) \arrow[r, "(\mathrm{K}_1^{\mathrm{Q}}(p_1),\mathrm{K}_1^{\mathrm{Q}}(p_2))", "\cong"'] & \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_1) \oplus \mathrm{K}_1^{\mathrm{Q}}(\mathcal{C}_2)\,. \end{tikzcd}$$

In particular, $\zeta_{\mathcal{C}_1 \times \mathcal{C}_2}$ is an isomorphism if and only if $\zeta_{\mathcal{C}_1}$ and $\zeta_{\mathcal{C}_2}$ are isomorphisms.

Denote by $\mathcal{A}_{\mathrm{ss}}$ the full subcategory of $\mathcal{A}$ consisting of all semisimple objects. Note that $\mathcal{A}_{\mathrm{ss}}$ is a Serre subcategory of $\mathcal{A}$, and hence $\mathcal{A}_{\mathrm{ss}}$ is itself an abelian category. Let $i\colon \mathcal{A}_{\mathrm{ss}} \hookrightarrow \mathcal{A}$ be the (exact) inclusion. Consider the commutative diagram,

$$\begin{tikzcd} \mathrm{K}_1^{\mathrm{B}}(\mathcal{A}_{\mathrm{ss}}) \arrow[r, "\mathrm{K}_1^{\mathrm{B}}(i)", "\cong"'] \arrow[d, "\zeta_{\mathcal{A}_{\mathrm{ss}}}"'] & \mathrm{K}_1^{\mathrm{B}}(\mathcal{A}) \arrow[d, "\zeta_{\mathcal{A}}"] \\ \mathrm{K}_1^{\mathrm{Q}}(\mathcal{A}_{\mathrm{ss}}) \arrow[r, "\mathrm{K}_1^{\mathrm{Q}}(i)", "\cong"'] & \mathrm{K}_1^{\mathrm{Q}}(\mathcal{A})\,. \end{tikzcd}$$

Since $\mathcal{A}$ is a length category, Bass' and Quillen's devissage theorems [6, VIII§3 thm. (3.4)(a)] and [15, §5 thm. 4] show that $\mathrm{K}_1^{\mathrm{B}}(i)$ and $\mathrm{K}_1^{\mathrm{Q}}(i)$ are isomorphisms. Hence, it suffices to argue that $\zeta_{\mathcal{A}_{\mathrm{ss}}}$ is an isomorphism. By assumption there is a finite set $\{S_1, \dots, S_n\}$ of representatives of the isomorphism classes of simple objects in $\mathcal{A}$. Note that every object A in $\mathcal{A}_{\mathrm{ss}}$ has unique decomposition $A = S_1^{a_1} \oplus \cdots \oplus S_n^{a_n}$ where $a_1, \dots, a_n \in \mathbb{N}_0$; we used here the assumption that A has finite length to conclude that the cardinal numbers a_i must be finite. Since one has $\mathrm{Hom}_{\mathcal{A}}(S_i, S_j) = 0$ for $i \neq j$, it follows that there is an equivalence of abelian categories,

$$\mathcal{A}_{\mathrm{ss}} \simeq (\mathsf{add}\, S_1) \times \cdots \times (\mathsf{add}\, S_n)\;.$$

Consider the ring $D_i = \mathrm{End}_{\mathcal{A}}(S_i)^{\mathrm{op}}$. As S_i is simple, Schur's lemma gives that D_i is a division ring. It easy to see that the functor $\mathrm{Hom}_{\mathcal{A}}(S_i, -)\colon \mathcal{A} \to \mathsf{Mod}\, D_i$ induces an equivalence $\mathsf{add}\, S_i \simeq \mathsf{proj}\, D_i$, and consequently one has an equivalence,

$$\mathcal{A}_{\mathrm{ss}} \simeq (\mathsf{proj}\, D_1) \times \cdots \times (\mathsf{proj}\, D_n)\;.$$

By Theorem (A.7) the maps $\zeta_{\mathsf{proj}\, D_1}, \dots, \zeta_{\mathsf{proj}\, D_n}$ are isomorphisms, so it follows from the equivalence above, and the general observation in the beginning af the proof, that $\zeta_{\mathcal{A}_{\mathrm{ss}}}$ is an isomorphism, as desired. □

Note that in this appendix, superscripts "C" (for classical), "B" (for Bass), and "Q" (for Quillen) are used to distinguish between various K-groups. Outside this appendix, K-groups without superscripts refer to Quillen's K-groups.

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